Some functional equations characterizing polynomials

By BARBARA KOCLĘGA-KULPA, TOMASZ SZOSTOK
and SZYMON WĄSOWICZ

Abstract. We present a method of solving functional equations of the type

$$F(x) - F(y) = (x - y)[b_1 f(\alpha_1 x + \beta_1 y) + \cdots + b_n f(\alpha_n x + \beta_n y)],$$

where $f, F : P \to P$ are unknown functions acting on an integral domain P and parameteres $b_1, \ldots, b_n; \alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n \in P$ are given. We prove that under some assumptions on the parameters involved, all solutions of such kind of equations are polynomials. We use this method to solve some concrete equations of this type. For example the equation

$$8[F(x) - F(y)] = (x - y)\left[f(x) + 3f\left(\frac{x+2y}{3}\right) + 3f\left(\frac{2x+y}{3}\right) + f(y)\right] \tag{1}$$

for $f, F : \mathbb{R} \to \mathbb{R}$ is solved without any regularity assumptions. It is worth noting that (1) stems from a well known quadrature rule used in numerical analysis.

Keywords functional equations on integral domain, quadrature rules, polynomial functions
MR(2000) Subject Classification 39B52, 39B22

1. Introduction

We deal with functional equations of some specific form. In the monograph of T. Riedel and P. Sahoo [8] the following equation stemming from Simpson quadrature rule

$$F(y) - F(x) = (y - x)\left[\frac{1}{6}f(x) + \frac{2}{3}f\left(\frac{x+y}{2}\right) + \frac{1}{6}f(y)\right]$$

is considered. We present a general solution of the generalized version of this equation i.e.

$$F(x) - F(y) = (x - y)[b_1 f(\alpha_1 x + \beta_1 y) + \cdots + b_n f(\alpha_n x + \beta_n y)].$$

In the last section we enumerate particular cases of this equation in order to show the usefulness of the obtained results.

We obtain it using the method based on the use of the lemma established by M. Sablik ([9, Lemma 2.3]) and improved by I. Pawlikowska (cf. [7]). We show at the end of the paper that our method is applicable also for some other functional equations.

In this paper by a *polynomial function of order* n we mean any solution of the functional equation $\Delta_h^{n+1} f(x) = 0$, where Δ_h^{n+1} stands for the $(n+1)$–th iterate of the difference operator $\Delta_h f(x) = f(x+h) - f(x)$. Observe that a continuous polynomial function of order n is a polynomial of degree at most n (see [5, Theorem 4, p. 398], or [6, Theorem 15.9.4, p. 452]).

Let us now quote Sablik's result. First we need some notations. Let G, H be Abelian groups and $SA^0(G,H) := H$, $SA^1(G,H) := \mathrm{Hom}(G,H)$ (i.e. the group of all homomorphisms from G into H), and for $i \in \mathbb{N}$, $i \geq 2$, let $SA^i(G,H)$ be the group of all i–additive and symmetric mappings from G^i into H. Furthermore, let $\mathscr{P} := \big\{(\alpha,\beta) \in \mathrm{Hom}(G,G)^2 : \alpha(G) \subset \beta(G)\big\}$. Finally, for $x \in G$ let $x^i = \underbrace{(x,\ldots,x)}_{i}$, $i \in \mathbb{N}$.

Lemma 1 *Fix $N \in \mathbb{N} \cup \{0\}$ and let $I_0, \ldots, I_N$ be finite subsets of $\mathscr{P}$. Suppose that H is uniquely divisible by $N!$ and let the functions $\varphi_i : G \to SA^i(G,H)$ and $\psi_{i,(\alpha,\beta)} : G \to SA^i(G,H)$ $((\alpha,\beta) \in I_i,\ ,\ i = 0,\ldots,N)$ satisfy*

$$\varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \varphi_i(x)(y^i) = \sum_{i=0}^{N} \sum_{(\alpha,\beta)\in I_i} \psi_{i,(\alpha,\beta)}\big(\alpha(x) + \beta(y)\big)(y^i) \qquad (2)$$

for every $x, y \in G$. Then φ_N is a polynomial function of order at most $k-1$, where

$$k = \sum_{i=0}^{N} \mathrm{card}\Big(\bigcup_{s=i}^{N} I_s\Big).$$

In the original Sablik's result the order of φ_N was not greater than k, while I. Pawlikowska noticed that it can be lowered by 1.

In our considerations we use Lemma 1 only for $G = H = \mathbb{R}$ and $N = 1$ and we consider only homomorphisms of the form $x \mapsto \alpha x$, $\alpha \in \mathbb{R}$. Consequently, let

$$\mathscr{P} := \big\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha = 0 \text{ or } \beta \neq 0\big\}$$

(the case $\alpha(\mathbb{R}) \not\subset \beta(\mathbb{R})$ may occur only if $\alpha \neq 0$, $\beta = 0$), in this setting Lemma 1 can be rephrased in the form

Lemma 2 *Let I_0, I_1 be finite subsets of $\mathscr{P}$. Let the functions $\varphi_i, \psi_{i,(\alpha,\beta)}$: $\mathbb{R} \to \mathbb{R}$ $((\alpha,\beta) \in I_i,\ i = 0,1)$ satisfy*

$$\varphi_1(x)y + \varphi_0(x) = \sum_{(\alpha,\beta)\in I_0} \psi_{0,(\alpha,\beta)}(\alpha x + \beta y) + y\Big[\sum_{(\gamma,\delta)\in I_1} \psi_{1,(\gamma,\delta)}(\gamma x + \delta y)\Big] \quad (3)$$

for any $x, y \in \mathbb{R}$. *Then* φ_1 *is a polynomial function of order at most equal to* $\mathrm{card}(I_0 \cup I_1) + \mathrm{card} I_1 - 1$.

2. Preliminaries

Lemma 3 *Let* P *be an integral domain with unit element* $\mathbb{1}$ *uniquely divisible by* 2 *such that* $n\mathbb{1} \neq 0$ *for all* $n \in \mathbb{N}$ *and let the functions* $f, F : P \to P$ *satisfy the equation*

$$F(x) - F(y) = (x - y)[b_1 f(\alpha_1 x + \beta_1 y) + \cdots + b_n f(\alpha_n x + \beta_n y)] \tag{4}$$

for all $x, y \in P$ *and some* $b_1, \ldots, b_n, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in P$. *Further, let* f *be a function of the form*

$$f(x) = a_k(x) + a_{k-1}(x) + \cdots + a_1(x) + a_0, \quad x \in P, \tag{5}$$

where $a_i, i \in \{1, \ldots, k\}$ *stands for a monomial function of order* i *and* $a_0 \in P$ *is some constant.*

Then each function $a_i, i \in \{1, \ldots, k\}$ *also satisfies* (4) *with some function* F_i.

Proof. We may assume $F(0) = 0$, so substituting $y = 0$ in (4) we obtain

$$F(x) = x[b_1 f(\alpha_1 x) + \cdots + b_n f(\alpha_n x)], \quad x \in P.$$

Using this formula in (4) we get

$$\begin{aligned} &x[b_1 f(\alpha_1 x) + \cdots + b_n f(\alpha_n x)] - y[b_1 f(\alpha_1 y) + \cdots + b_n f(\alpha_n y)] = \\ &(x - y)[b_1 f(\alpha_1 x + \beta_1 y) + \cdots + b_n f(\alpha_n x + \beta_n y)], \quad x, y \in P. \end{aligned} \tag{6}$$

It is easy to see that if f, g are solutions to this equation, then also $f + g$ satisfies this equation. Moreover, if f satisfies (6) and $\alpha \in P$ is given, then also αf is a solution. Now substitute ax, ay in place of x and y, We obtain (after cancelling a on both sides) that

$$x[b_1 f(\alpha_1 ax) + \cdots + b_n f(\alpha_n ax)] - y[b_1 f(\alpha_1 ay) + \cdots + b_n f(\alpha_n ay)] =$$

$$(x - y)[b_1 f(\alpha_1 ax + \beta_1 ay) + \cdots + b_n f(\alpha_n ax + \beta_n ay)], \quad x, y \in P,$$

which means that the function $x \mapsto f(ax)$ also satisfies our equation (if f does).

Now consider f of the form (5), then g_1 given by

$$g_1(x) := 2^k f(x) - f(2x) = (2^k - 2^{k-1})a_{k-1}(x) + (2^k - 2^{k-2})a_{k-2}(x)$$

$$+ \cdots + (2^k - 1)a_0, \quad x \in P.$$

satisfies (6). Thus we may now put $g_2(x) := 2^{k-1}g_1(x) - g_1(2x), \quad x \in P$. The function g_2 is clearly of the form

$$g_2(x) = A^2_{k-2}a_{k-2}(x) + \cdots + A^2_0 a_0, \quad x \in P$$

for some numbers $A^2_i, i = 0, \ldots, k-2$. Repeating this procedure finitely many times, we obtain that

$$g_{k-1}(x) = A^{k-1}_1 a_1(x) + A^{k-1}_0 a_0, \quad x \in P$$

satisfies the equation (6). However since every constant is a solution of this equation, a_1 has to satisfy it. But

$$g_{k-2}(x) = A^{k-2}_2 a_2(x) + A^{k-2}_1 a_1(x) + A^{k-2}_0 a_0, \quad x \in P.$$

The functions g_{k-2} and $x \mapsto A^{k-2}_1 a_1(x) + A^{k-2}_0 a_0$, satisfy (6), thus also a_2 is a solution of (6). In the same way one can show that every a_i, $i = 1, 2, \ldots, k$ satisfy the equation (6), and consequently, also the equation (4) (with $F_i(x) = x[b_1 a_i(\alpha_1 x) + \cdots + b_n a_i(\alpha_n x)]$).

Lemma 4 *Let P be an integral domain with unit element $\mathbb{1}$ such that $\mathbb{1}+\mathbb{1} \neq 0$ and $\mathbb{1}+\mathbb{1}+\mathbb{1} \neq 0$, let $A_3 : P^3 \to P$ be a symmetric and $3-$additive function. Further, let $a_3 : P \to P$ be the diagonalization of A_3 satisfying (4) for all $x, y \in P$ and some $b_1, \ldots, b_n$, $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n \in P$ such that for all $i = 1, \ldots, n$ and $x, y, z \in P$ we have*

$$A_3(\alpha_i x, y, z) = \alpha_i A_3(x, y, z)$$

and

$$A_3(\beta_i x, y, z) = \beta_i A_3(x, y, z).$$

If $\xi, \eta, \gamma, \delta \in P$ are defined as follows

$$\begin{aligned} \xi &:= b_1\alpha_1\beta_1^2 + \cdots + b_n\alpha_n\beta_n^2, \\ \eta &:= b_1\alpha_1^2\beta_1 + \cdots + b_n\alpha_n^2\beta_n, \\ \gamma &:= b_1\alpha_1^3 + \cdots + b_n\alpha_n^3, \\ \delta &:= b_1\beta_1^3 + \cdots + b_n\beta_n^3, \end{aligned} \tag{7}$$

then the following assertions hold true:

(i) *if* $\gamma \neq \delta$, *then* $a_3 = 0$;

(ii) *if* $\gamma = \delta$ *and* $\xi \neq \eta$, *then* $a_3 = 0$;

(iii) *if* $\gamma = \delta$ *and* $\xi = \eta \neq 0$, *then* $a_3(x) = ax^3$ *for some* $a \in P$ *and all* $x \in P$ *and if* $a \neq 0$, *then* $\delta = 3\xi$.

(iv) *if* $\gamma = \delta \neq 0$ *and* $\xi = \eta = 0$, *then* $a_3 = 0$.

(v) *if* $\gamma = \delta = \xi = \eta = 0$, *then* a_3 *may be any function of the assumed form.*

Conversely, in each of the above cases the function a_3 *with the function* F *given by the formula*

$$F(x) = x[b_1 a_3(\alpha_1 x) + \cdots + b_n a_3(\alpha_n x)] + c, \quad x \in P \tag{8}$$

where $c \in P$ *is some constant, is a solution of* (4).

Proof. Let us rewrite the equation (4) for $f = a_3$

$$F(x) - F(y) = (x-y)[b_1 a_3(\alpha_1 x + \beta_1 y) + ... + b_n a_3(\alpha_n x + \beta_n y)], \quad x, y \in P. \tag{9}$$

Assuming without loss of generality $F(0) = 0$, take here $y = 0$ and substitute the obtained formula to (9)

$$\begin{aligned} &x\,[b_1 a_3(\alpha_1 x) + ... + b_n a_3(\alpha_n x) - b_1 a_3(\alpha_1 x + \beta_1 y) - ... - b_n a_3(\alpha_n x + \beta_n y)] = \\ &y\,[b_1 a_3(\alpha_1 y) + ... + b_n a_3(\alpha_n y) - b_1 a_3(\alpha_1 x + \beta_1 y) - ... - b_n a_3(\alpha_n x + \beta_n y)]. \end{aligned}$$

Using here the equality $a_3(x) = A_3(x, x, x)$ after some calculations we get for all $x, y \in P$

$$\begin{aligned} 3(b_1\alpha_1\beta_1^2 + ... + b_n\alpha_n\beta_n^2)&(y - x)A_3(x, y, y) + \\ &+ 3(b_1\alpha_1^2\beta_1 + ... + b_n\alpha_n^2\beta_n)(y - x)A_3(x, x, y) = \\ &= (b_1\alpha_1^3 + ... + b_n\alpha_n^3)y[a_3(y) - a_3(x)] + \\ &+ (b_1\beta_1^3 + ... + b_n\beta_n^3)(x - y)a_3(y). \end{aligned}$$

In view of the notations from (7) we have

$$3(y - x)[\xi A_3(x, y, y) + \eta A_3(x, x, y)] = [\delta x + (\gamma - \delta)y]a_3(y) - \gamma y a_3(x). \tag{10}$$

Now substitute in (10) $-y$ in place of y. Then

$$-3(x+y)[\xi A_3(x, y, y) - \eta A_3(x, x, y)] = -[\delta x - (\gamma - \delta)y]a_3(y) + \gamma y a_3(x). \tag{11}$$

Adding the equations (10) and (11) we arrive at

$$3\xi[(y-x)-(x+y)]A_3(x,y,y)+3\eta[(y-x)+(x+y)]A_3(x,x,y) = 2y[\gamma-\delta]a_3(y),$$

i.e.

$$-3\xi x A_3(x,y,y) + 3\eta y A_3(x,x,y) = (\gamma-\delta)y a_3(y), \quad x,y \in P. \tag{12}$$

Further, in (12) we put $2x$ in place of x. Thus

$$-12\xi x A_3(x,y,y) + 12\eta y A_3(x,x,y) = (\gamma-\delta)y a_3(y), \quad x,y \in P. \tag{13}$$

Multiplying (12) by 4 we have

$$-12\xi x A_3(x,y,y) + 12\eta y A_3(x,x,y) = 4(\gamma-\delta)y a_3(y), \quad x,y \in P \tag{14}$$

After comparing equations (14) and (13) we observe that

$$(\gamma-\delta)a_3(y) = 0, \quad y \neq 0.$$

Clearly, this equation is also satified for $y = 0$.

Now if $\gamma \neq \delta$, then $a_3(y) = 0$ for all $y \in P$ and (i) is proved.

Assume now that $\gamma = \delta$. Then equation (12) takes form

$$\xi x A_3(x,y,y) = \eta y A_3(x,x,y), \quad x,y \in P. \tag{15}$$

Take here $y = x$

$$(\xi-\eta)a_3(x) = 0, \quad x \in P, \quad x,y \in P.$$

Similarly as above, if $\xi \neq \eta$, then $a_3(x) = 0, \quad x \in P$ and (ii) holds.

This means that we may assume that $\xi = \eta$. Let us now consider the case $\xi = \eta = 0$. Then we have by (10)

$$\gamma x a_3(y) = \gamma y a_3(x), \quad x,y \in P.$$

Taking here $y = 2x$ we get

$$\gamma a_3(x) = 0, \quad x \in P, x \neq 0.$$

If $\gamma \neq 0$, then $a_3(x) = 0$ for $x \in P$ and (iv) holds. If, on the other hand, $\gamma = 0$, then

$$\gamma = \delta = \eta = \xi = 0.$$

In this case the equation (10) reduces to a trivial one, so (v) holds.

This means that from this point we may assume that $\gamma = \delta$ and $\xi = \eta \neq 0$ (as in (iii)). Using these facts in (15) we obtain

$$xA_3(x, y, y) = yA_3(x, x, y), \quad x, y \in P. \tag{16}$$

Taking here $x + y$ in place of x we may write

$$(x - y)A_3(x, y, y) - yA_3(x, x, y) = -xa_3(y) \;\; x, y \in P.$$

Thus in view of (16) we obtain

$$xa_3(y) = yA_3(x, y, y), \quad x, y \in P. \tag{17}$$

Interchanging in this equation x with y we arrive at

$$ya_3(x) = xA_3(x, x, y), \quad x, y \in P. \tag{18}$$

Now, multiply (17) by x, and (18) by y^2 to get

$$x^2a_3(y) = xyA_3(x, y, y), \quad x, y \in P \tag{19}$$

and

$$y^3a_3(x) = xy^2A_3(x, x, y), \quad x, y \in P. \tag{20}$$

On the other hand, multiply (16) by y and use (19)

$$x^2a_3(y) = y^2A_3(x, x, y),$$

which multiplied by x gives us

$$x^3a_3(y) = xy^2A_3(x, x, y), \quad x, y \in P.$$

Comparing this equation with (20) we obtain

$$x^3a_3(y) = y^3a_3(x), \quad x, y \in P,$$

which after substituting $y = \mathbb{1}$ yields

$$a_3(x) = a_3(\mathbb{1})x^3, \quad x \in P. \tag{21}$$

Now it is enough to show that if $a_3(\mathbb{1}) \neq 0$, then $\delta = 3\xi$. To this end use (21) in (17)

$$a_3(\mathbb{1})xy^3 = yA_3(x, y, y)$$

i.e.

$$A_3(x, y, y) = a_3(\mathbb{1})xy^2, \quad x, y \in P \tag{22}$$

and by symmetry of A_3

$$A_3(x,x,y) = a_3(\mathbb{1})x^2y, \quad x,y \in P. \tag{23}$$

By (22) and (23) and using the equalities $\gamma = \delta$ and $\eta = \xi$ in (10) we get for all $x, y \in P$

$$3\xi(y-x)a_3(\mathbb{1})(xy^2+x^2y) = \delta a_3(\mathbb{1})[xy^3 - x^3y], \quad x,y \in P.$$

Thus $3\xi = \delta$, which proves (iii).

On the other hand, it is easy to show that all functions of the forms (i)-(v) satisfy equation (10) (and (9) with F given by (8)).

Lemma 5 *Let P be an integral domain with a unit element $\mathbb{1}$ such that $\mathbb{1}+\mathbb{1} \neq 0$ and let $A_2 : P^2 \to P$ be a symmetric and $2-$additive function. Further, let $a_2 : P \to P$ be the diagonalization of A_2 satisfying the equation (4) for all $x, y \in P$ and some $b_1, \dots, b_n$, $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_n \in P$ such that for all $i = 1, \dots, n$ and $x, y \in P$ we have*

$$A_2(\alpha_i x, y) = \alpha_i A_2(x,y)$$

and

$$A_2(\beta_i x, y) = \beta_i A_2(x,y).$$

If $\gamma, \delta, \eta \in P$ are defined as follows

$$\begin{array}{l} \gamma := b_1\alpha_1^2 + \cdots + b_n\alpha_n^2, \\ \delta := b_1\beta_1^2 + \cdots + b_n\beta_n^2, \\ \eta := b_1\alpha_1\beta_1 + \cdots + b_n\alpha_n\beta_n, \end{array} \tag{24}$$

then the following assertions hold true:

(i) *if $\gamma \neq \delta$, then $a_2 = 0$;*

(ii) *if $\gamma = \delta$ and $\gamma + 2\eta \neq 0$, then $a_2(x) = ax^2$ for some $a \in P$ and all $x \in P$;*

(iii) *if $\gamma = \delta$, $\gamma + 2\eta = 0$ and $a_2 \neq 0$, then $\gamma = \delta = \eta = 0$ and a_2 may be any function.*

Conversely, in each of the above cases the function a_2 with the function F given by the formula

$$F(x) = x[b_1a_2(\alpha_1 x) + \cdots + b_na_2(\alpha_n x)] + c, \quad x \in P \tag{25}$$

where $c \in P$ is some constant, is a solution of (4).

Proof. Similarly, as in the proof of Lemma 4 from (4) we obtain

$$x\,[b_1a_2(\alpha_1x)+...+b_na_2(\alpha_nx)-b_1a_2(\alpha_1x+\beta_1y)-...-b_na_2(\alpha_nx+\beta_ny)]=$$
$$y\,[b_1a_2(\alpha_1y)+...+b_na_2(\alpha_ny)-b_1a_2(\alpha_1x+\beta_1y)-...-b_na_2(\alpha_nx+\beta_ny)].$$

Thus using here the form of a_2 we get

$$-x\,[(b_1\beta_1^2+...+b_n\beta_n^2)a_2(y)+2(b_1\alpha_1\beta_1+...+b_n\alpha_n\beta_n)A_2(x,y)]=$$
$$=y\,[(b_1\alpha_1^2+...+b_n\alpha_n^2-b_1\beta_1^2-...-b_n\beta_n^2)a_2(y)-(b_1\alpha_1^2+...+b_n\alpha_n^2)a_2(x)+$$
$$-\;\;2(b_1\alpha_1\beta_1+...+b_n\alpha_n\beta_n)A_2(x,y)],$$

or, equivalently,

$$\gamma y[a_2(x)-a_2(y)]-\delta(x-y)a_2(y)=2\eta(x-y)A_2(x,y),\ \ x,y\in P. \tag{26}$$

Interchanging in this equation x with y we have

$$\gamma x[a_2(x)-a_2(y)]-\delta(x-y)a_2(x)=2\eta(x-y)A_2(x,y),\ \ x,y\in P. \tag{27}$$

Comparing the equations (27) and (26) we are able to write

$$(\gamma-\delta)(x-y)[a_2(x)-a_2(y)]=0$$

for all $x,y\in P$.

Now substitute $y=0$, then since $a_2(0)=0$

$$(\gamma-\delta)a_2(x)=0,\ x\neq 0,$$

which is clearly satisfied also for $x=0$.

Now we consider two cases:

1° $\gamma\neq\delta$

In this case we obtain $a_2(x)=0$ for every $x\in P$, which proves (i).

2° $\gamma=\delta$

We have from (26)

$$\gamma[ya_2(x)-xa_2(y)]=2\eta(x-y)A_2(x,y),\ \ x,y\in P. \tag{28}$$

If we substitute here $x+y$ and $x-y$ in place of x and y, respectively, then after some calculations we arrive at

$$a_2(x)(2\gamma y+4\eta y)+a_2(y)(2\gamma y-4\eta y)=4\gamma xA_2(x,y),$$

thus

$$(\gamma+2\eta)ya_2(x)+(\gamma-2\eta)ya_2(y)=2\gamma xA_2(x,y), \quad x,y\in P. \tag{29}$$

Now interchange x with y. We have

$$(\gamma+2\eta)xa_2(y)+(\gamma-2\eta)xa_2(x)=2\gamma yA_2(x,y), \quad x,y\in P. \tag{30}$$

Multiplying (29) by y and (30) by x and comparing the obtained equations we may write

$$(\gamma+2\eta)y^2a_2(x)+(\gamma-2\eta)y^2a_2(y)=(\gamma+2\eta)x^2a_2(y)+(\gamma-2\eta)x^2a_2(x),$$

which gives us

$$a_2(x)[y^2(\gamma+2\eta)-x^2(\gamma-2\eta)]=a_2(y)[x^2(\gamma+2\eta)-y^2(\gamma-2\eta)], \quad x,y\in P.$$

Substituting in this equation $y=\mathbb{1}$ we have

$$a_2(x)[(\gamma+2\eta)-x^2(\gamma-2\eta)]=a_2(\mathbb{1})[x^2(\gamma+2\eta)-(\gamma-2\eta)], \quad x\in P. \tag{31}$$

Now we are going to show that if $a_2\neq 0$, then $\gamma-2\eta=0$. To this end put in (28) $y=2x$

$$\gamma[2xa_2(x)-xa_2(2x)]=-2\eta xA_2(x,2x),$$

i.e.

$$(\gamma-2\eta)a_2(x)=0, \quad x\in P, x\neq 0.$$

Consequently,

$$\gamma-2\eta=0 \tag{32}$$

and using this equality in (31) we obtain

$$(\gamma+2\eta)a_2(x)=x^2(\gamma+2\eta)a_2(\mathbb{1}), \quad x\in P. \tag{33}$$

Consider the case $\gamma+2\eta\neq 0$, then

$$a_2(x)=a_2(\mathbb{1})x^2, \quad x\in P,$$

which proves (ii). On the other hand, if $\gamma+2\eta=0$, then together with (32) we have $\gamma=\eta=0$ and the equation (33) is trivial, which gives (iii).

To prove the converse assertion it is enough to use the obtained forms of a_2 and F given by (8) and check that the equation (4) is satisfied.

Lemma 6 *Let P be an integral domain and let $a_1 : P \to P$ be an additive function satisfying equation* (4) *for all $x, y \in P$ and some $b_1, \dots, b_n$, $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_n \in P$ such that for all $i = 1, \dots, n$ and $x \in P$ we have*

$$a_1(\alpha_i x) = \alpha_i a_1(x)$$

and

$$a_1(\beta_i x) = \beta_i a_1(x).$$

If $\gamma, \delta \in P$ are defined as follows

$$\begin{aligned} \gamma &:= b_1\alpha_1 + \dots + b_n\alpha_n, \\ \delta &:= b_1\beta_1 + \dots + b_n\beta_n, \end{aligned} \tag{34}$$

then the following assertions hold true:

(i) *if $\gamma \neq \delta$, then $a_1 = 0$;*

(ii) *if $\gamma = \delta \neq 0$, then $a_1(x) = ax$ for some $a \in P$ and all $x \in P$.*

(iii) *if $\gamma = \delta = 0$, then a_1 may be any function.*

Conversely, in each of the above cases function a_1 with

$$F(x) = x[b_1a_1(\alpha_1 x) + \dots + b_na_1(\alpha_n x)] + c \tag{35}$$

where $c \in P$ is some constant, is a solution of (4).

Proof. Similarly as in the proof of Lemma 4 and Lemma 5 from (4) we obtain

$$\begin{aligned} &x\,[b_1a_1(\alpha_1x) + ... + b_na_1(\alpha_nx) - b_1a_1(\alpha_1x + \beta_1y) - ... - b_na_1(\alpha_nx + \beta_ny)] = \\ &y\,[b_1a_1(\alpha_1y) + ... + b_na_1(\alpha_ny) - b_1a_1(\alpha_1x + \beta_1y) - ... - b_na_1(\alpha_nx + \beta_ny)]. \end{aligned}$$

Thus using here additivity of a_1 and in view of the notations (34) we get

$$\gamma y a_1(x) = a_1(y)[(\gamma - \delta)y + \delta x], \quad x, y \in P. \tag{36}$$

If we substitute in (36) $x + y$ in place of x, then after some calculations we may write

$$\gamma y a_1(x) = \delta x a_1(y), \quad x, y \in P. \tag{37}$$

Comparing the obtained equation and the equation (36) we get

$$(\gamma - \delta) y a_1(y) = 0, \quad y \in P.$$

Now we consider two cases:

1° $\gamma \neq \delta$

In this case we obtain $a_1(x) = 0$ for every $x \in P$, which gives (i).

2° $\gamma = \delta$

We have from (37)

$$\gamma y a_1(x) = \gamma x a_1(y), \quad x, y \in P.$$

Substituting in this equation $y = \mathbb{1}$ we obtain

$$\gamma a_1(x) = \gamma a_1(\mathbb{1})x, \quad x \in P.$$

If $\gamma \neq 0$, then $a_1(x) = a_1(\mathbb{1})x$ for $x \in P$ and (ii) holds. On the other hand, if $\gamma = 0$, then $\delta = 0$ and our equation reduces to a trivial one.

Now, it is enough to show that all the functions of the forms (i)-(iii) satisfy equation (4) with F given by (35).

3. Results

Now we are in position to use the above lemmas to solve some equations. We start with the main result of the paper.

Theorem 1 *If the functions $F, f : \mathbb{R} \to \mathbb{R}$ satisfy the equation*

$$8[F(y) - F(x)] = (y - x)\left[f(x) + 3f\left(\frac{x+2y}{3}\right) + 3f\left(\frac{2x+y}{3}\right) + f(y)\right] \tag{38}$$

for all $x, y \in \mathbb{R}$, then there exist $a, b, c, d, e \in \mathbb{R}$, such that

$$f(x) = ax^3 + bx^2 + cx + d, \quad x \in \mathbb{R}$$

and

$$F(x) = \frac{1}{4}ax^4 + \frac{1}{3}bx^3 + \frac{1}{2}cx^2 + dx + e, \quad x \in \mathbb{R}.$$

On the other hand, the functions f and F given by the above formulas satisfy (38)*.*

Proof. Assume that F, f satisfy (38). Put $x+y$ instead of y in this equation. Then we get

$$8F(x+y)-8F(x)=yf(x)+3yf\left(\frac{3x+2y}{3}\right)+3yf\left(\frac{3x+y}{3}\right)+yf(x+y),$$

i.e.

$$8F(x)+yf(x)=8F(x+y)-3yf\left(\frac{3x+2y}{3}\right)-3yf\left(\frac{3x+y}{3}\right)-yf(x+y).$$

Take in Lemma 2

$$I_0=\{(1,1)\},\quad I_1=\left\{\left(1,\frac{2}{3}\right),\left(1,\frac{1}{3}\right),(1,1)\right\}.$$

Then f is a polynomial function of order at most 5. Thus

$$f(x)=a_0+a_1(x)+a_2(x)+a_3(x)+a_4(x)+a_5(x),\quad x\in\mathbb{R},$$

where $a_i:\mathbb{R}\to\mathbb{R}$ is a diagonalization of a symmetric $i-$additive function $A_i:\mathbb{R}^i\to\mathbb{R}$, i.e. $a_i(x)=A_i(\underbrace{x,...,x}_{i})$, $x\in\mathbb{R}$. Moreover, from Lemma 3 we know that every function a_i, $i\in\{1,\dots,5\}$ satisfies (38).

Substitute $x=0$ in (38). We obtain

$$8F(y)=y\left[f(0)+3f\left(\frac{2y}{3}\right)+3f\left(\frac{y}{3}\right)+f(y)\right]+8F(0),\quad y\in\mathbb{R},$$

which used in (38) gives for all $x,y\in\mathbb{R}$

$$\begin{aligned}y&\left[f(0)+3f\left(\frac{2y}{3}\right)+3f\left(\frac{y}{3}\right)+f(y)\right]-x\left[f(0)+3f\left(\frac{2x}{3}\right)+3f\left(\frac{x}{3}\right)+f(x)\right]=\\&=(y-x)\left[f(x)+3f\left(\frac{x+2y}{3}\right)+3f\left(\frac{2x+y}{3}\right)+f(y)\right],\end{aligned}$$

or, equivalently,

$$\begin{aligned}y&\left[f(0)+3f\left(\frac{2y}{3}\right)+3f\left(\frac{y}{3}\right)-f(x)-3f\left(\frac{x+2y}{3}\right)-3f\left(\frac{2x+y}{3}\right)\right]=\\&=x\left[f(0)+3f\left(\frac{2x}{3}\right)+3f\left(\frac{x}{3}\right)-f(y)-3f\left(\frac{x+2y}{3}\right)-3f\left(\frac{2x+y}{3}\right)\right].\end{aligned}$$

Substituting here $3x$ and $3y$ in place of x and y, respectively, we may write

$$\begin{aligned}y[f(0)\,-\,&f(3x)+3f(2y)+3f(y)-3f(x+2y)-3f(2x+y)]=\\&=x[f(0)-f(3y)+3f(2x)+3f(x)-3f(x+2y)-3f(2x+y)].\end{aligned}\tag{39}$$

Further, take $y = 2x$ in (39). Then

$$f(6x) - 3f(5x) + 3f(4x) - 2f(3x) + 3f(2x) - 3f(x) + f(0) = 0, \quad x \neq 0, \quad (40)$$

which is also true for $x = 0$.

Since a_5 satisfies (38), then it also satisfies (40) and using 5−additivity of a_5 we get

$$(6^5 - 3 \cdot 5^5 + 3 \cdot 4^5 - 2 \cdot 3^5 + 3 \cdot 2^5 - 3)a_5(x) = 0, \quad x \in \mathbb{R},$$

i.e.

$$1080 a_5(x) = 0, \quad x \in \mathbb{R}.$$

This means that $a_5(x) = 0$ for $x \in \mathbb{R}$. Similarly, we show that $a_4(x) = 0$ for all $x \in \mathbb{R}$. Indeed, equation (40) for a_4 takes form

$$72 a_4(x) = 0, \quad x \in \mathbb{R}.$$

Now by Lemmas 4, 5, 6 we obtain that

$$f(x) = ax^3 + bx^2 + cx + d$$

for some $a, b, c, d \in \mathbb{R}$ and all $x \in \mathbb{R}$. Moreover, by (39) we obtain the desired formula for F. To finish the proof it sufficies to check that these function satisfy our equation.

Remark 1 For the sake of simplicity we proved the above thereom for functions acting on reals. However, it is easy to see that the same proof may be applied for functions acting on an integral domain P, uniquely divisible by 5! and such that $n\mathbb{1} \neq 0$ for all $n \in \mathbb{N}$, where $\mathbb{1}$ is a unit element of P.

Let us point out that our method can be applied to solve some other functional equations. We give an example (see [4]).

Theorem 2 *The functions $f, F : \mathbb{R} \to \mathbb{R}$ satisfy*

$$F(y) - F(x) = (y - x)\left[\frac{1}{6}f(x) + \frac{2}{3}f\left(\frac{x+y}{2}\right) + \frac{1}{6}f(y)\right], \quad x, y \in \mathbb{R} \quad (41)$$

if and only if

$$f(x) = ax^3 + bx^2 + cx + d, \quad x \in \mathbb{R}$$

and

$$F(x) = \frac{1}{4}ax^4 + \frac{1}{3}bx^3 + \frac{1}{2}cx^2 + dx + e, \quad x \in \mathbb{R},$$

where $a, b, c, d, e \in \mathbb{R}$ are some numbers.

Proof. Substitute $y = y + x$ in (41), then we have

$$-\frac{1}{6}yf(x) - F(x) = -F(x+y) + y\left[\frac{2}{3}f\left(x+\frac{y}{2}\right) + \frac{1}{6}f(x+y)\right], \quad x, y \in \mathbb{R}.$$

From Lemma 2 we know that f must be a polynomial function of order at most 3. Using Lemmas 3, 4, 5 and 6 we obtain that f must be an ordinary polynomial and putting in (41) $x = 0$ we obtain the desired form of F.

On the other hand, it is easy to show that f and F of this form satisfy (41).

Some geometric problems considered recently by C. Alsina, M. Sablik and J. Sikorska in [2] lead to the functional equation

$$2F(y) - 2F(x) = (y-x)\left[f\left(\frac{x+y}{2}\right) + \frac{f(x)+f(y)}{2}\right],$$

solved also by B. Ebanks and M. Sablik (see [10]). Further, in [3] Sh. Haruki considered the equation

$$f(x) - f(y) = (x-y)\left(\frac{g(x)+g(y)}{2}\right)$$

and on the other hand, J. Aczél in [1] solved the equation

$$f(x) - f(y) = (x-y)g(x+y).$$

It is easy to observe that also all these equations may be solved with use of our method.

References

[1] J. Aczél, A mean value property of the derivative of quadratic polynomials—without mean values and derivatives, Math. Mag. 58, no. 1, 42–45 (1985)

[2] C. Alsina, M. Sablik, J. Sikorska,: On a functional equation based upon a result of Gaspard Monge, J. Geom. 85, no. 1-2, 1–6 (2006)

[3] Sh. Haruki, A property of quadratic polynomials, Amer. Math. Monthly 86 no. 7, 577–579, (1979)

[4] B. Koclęga–Kulpa, T. Szostok,: On some equations connected to Hadamard inequalities, Aequationes Math. 75 (2008), 119–129.

[5] M. Kuczma,: An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality, Państwowe Wydawnictwo Naukowe (Polish Scientific Publishers) and Uniwersytet Śląski, Warszawa–Kraków–Katowice (1985)

[6] M. Kuczma,: An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality. Second edition. Birkhäuser Verlag, Basel (2009)

[7] I. Pawlikowska,: Solutions of two functional equations using a result of M. Sablik, Aequationes Math. 72, 177–190 (2006)

[8] T. Riedel, P. K. Sahoo,: Mean value theorems and functional equations, World Scientific, Singapore–New Jersey–London–Hong Kong (1998)

[9] M. Sablik,: Taylor's theorem and functional equations, Aequationes Math. 60, 258–267 (2000)

[10] Report of Meeting. The Fifth Katowice-Debrecen Winter Seminar on Functional Equations and Inequalities, Ann. Math. Sil. no. 19, 65–78 (2005)

Barbara Koclęga-Kulpa
Institute of Mathematics,
Silesian University,
Bankowa 14,
40-007 Katowice,
Poland
E-mail: koclega@ux2.math.us.edu.pl

Tomasz Szostok
Institute of Mathematics,
Silesian University,
Bankowa 14,
40-007 Katowice,
Poland
E-mail: szostok@ux2.math.us.edu.pl

Szymon Wąsowicz
Department of Mathematics and Computer Science
University of Bielsko–Biała
Willowa 2,
43-309 Bielsko–Biała,
Poland
E-mail: swasowicz@ath.bielsko.pl