Published:
C. Bandle et al. (eds.), *Inequalities and Applications 2010*,
International Series of Numerical Mathematics 161, pp. 175–182, © Springer Basel 2012.
DOI 10.1007/978-3-0348-0249-9_13

FUNCTIONS GENERATING STRONGLY SCHUR–CONVEX SUMS

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Dedicated to the Memory of Professor Wolfgang Walter

ABSTRACT. The notion of strongly Schur–convex functions is introduced and functions generating strongly Schur–convex sums are investigated. The results presented are counterparts of the classical Hardy–Littlewood–Pólya majorization theorem and the theorem of Ng characterizing functions generating Schur–convex sums. It is proved, among others, that for some (for every) $n \geqslant 2$, the function $F(x_1, \ldots, x_n) = f(x_1) + \cdots + f(x_n)$ is strongly Schur–convex with modulus c if and only if f is of the form $f(x) = g(x) + a(x) + c\|x\|^2$, where g is convex and a is additive.

1. INTRODUCTION

Let $\mathcal{I} \subset \mathbb{R}$ be an interval and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathcal{I}^n$, where $n \geqslant 2$ (throughout the paper n will be always assumed to be a natural number). Following Schur [14] (cf. also [6]) we say that x *is majorized by* y, and write $x \preccurlyeq y$, if there exists a doubly stochastic $n \times n$ matrix P (i.e. matrix containing nonnegative elements with all rows and columns summing up to 1) such that $x = y \cdot P$. A function $F : \mathcal{I}^n \to \mathbb{R}$ is said to be *Schur–convex* if $F(x) \leqslant F(y)$ whenever $x \preccurlyeq y,\ x, y \in \mathcal{I}^n$.

It is known, by the classical works of Schur [14], Hardy–Littlewood–Pólya [2] and Karamata [4] that if a function $f : \mathcal{I} \to \mathbb{R}$ is convex then it *generates Schur–convex sums*, that is the function $F : \mathcal{I}^n \to \mathbb{R}$ defined by

$$F(x) = F(x_1, \ldots, x_n) = f(x_1) + \cdots + f(x_n)$$

is Schur–convex. It is also known that the convexity of f is a sufficient but not necessary condition under which F is Schur–convex. In 1987 C. T. Ng [9] gave a full characterization of functions (defined on a convex and open subset of $\mathbb{R}^n$) generating Schur–convex sums.

In this note we introduce the notion of strong Schur–convexity (in X^n, where X is an inner product space) and we present a counterpart of the Ng representation theorem for functions generating strongly Schur–convex sums.

Let $\big(X, \|\cdot\|\big)$ be a (real) inner product space. We consider the space X^n $(n \geqslant 2)$ with the product norm

$$|\!|\!|x|\!|\!| = \sqrt{\|x_1\|^2 + \cdots + \|x_n\|^2}, \quad x = (x_1, \ldots, x_n) \in X^n.$$

Date: July 12, 2012.

2000 *Mathematics Subject Classification.* Primary: 26B25. Secondary: 39B62.

Key words and phrases. Schur–convex functions, strongly Schur–convex functions, strongly convex functions, strongly Jensen–convex functions, strongly Wright–convex functions, doubly stochastic matrices, majorization.

The research of the third author was supported by the University of Bielsko–Biała internal grant no. 08/II/GW/2009.

Similarly as in the classical case we define the majorization in X^n. Namely, given two n–tuples $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)\in X^n$ we say that x *is majorized by* y, written $x\preccurlyeq y$, if

$$(x_1,\dots,x_n)=(y_1,\dots,y_n)\cdot P$$

for some doubly stochastic $n\times n$ matrix P.

Note that if $x\preccurlyeq y$ then $|\!|\!|x|\!|\!|^2\leqslant |\!|\!|y|\!|\!|^2$. It follows, for instance, from the fact that the function $\|\cdot\|^2: X\to\mathbb{R}$ is convex and so it generates Schur–convex sums (the proof is exactly the same as in the classical case of $X=\mathbb{R}$; cf. also the proof of Theorem 1 below, where we repeat the argument for the sake of completeness).

Let D be a convex subset of X and let $c>0$. Recall that a function $f: D\to\mathbb{R}$ is called *strongly convex with modulus* c (cf. e.g. [3], [11], [13]) if

$$f\big(tx+(1-t)y\big)\leqslant tf(x)+(1-t)f(y)-ct(1-t)\|x-y\|^2 \tag{1}$$

for all $x,y\in D$ and $t\in[0,1]$; f is called *strongly Jensen–convex with modulus* c if condition (1) is assumed only for $t=\frac{1}{2}$, that is

$$f\Big(\frac{x+y}{2}\Big)\leqslant\frac{f(x)+f(y)}{2}-\frac{c}{4}\|x-y\|^2,\quad x,y\in D.$$

Recall also that f is said to be *strongly Wright–convex with modulus* c (see [8]) if

$$f\big(tx+(1-t)y\big)+f\big((1-t)x+ty\big)\leqslant f(x)+f(y)-2ct(1-t)\|x-y\|^2$$

for all $x,y\in D$ and $t\in[0,1]$.

The concept of strong convexity was inspired by the optimization theory and many properties and applications of it can be found in the literature. For some recent results concerning strongly (Jensen–, Wright–) convex functions the reader is referred to [1], [7], [8], [10], [12].

Motivated by the above definitions we propose a strengthening of the notion of Schur–convexity. Let D be a convex subset of X, $c>0$ and $n\geqslant 2$. We say that a function $F: D^n\to\mathbb{R}$ is *strongly Schur–convex with modulus* c if

$$x\preccurlyeq y\quad\Longrightarrow\quad F(x)\leqslant F(y)-c\big(|\!|\!|y|\!|\!|^2-|\!|\!|x|\!|\!|^2\big)$$

for all $x,y\in D$. Note that the usual Schur–convexity corresponds to the case $c=0$.

2. Schur–convex sums

In this section we prove that strongly convex functions generate strongly Schur–convex sums and functions generating strongly Schur–convex sums are strongly Jensen–convex (and hence, under some regularity assumptions they are strongly convex).

Theorem 1. *Let D be a convex subset of an inner product space $\big(X,\|\cdot\|\big)$ and $c>0$. If a function $f: D\to\mathbb{R}$ is strongly convex with modulus c, then for every $n\geqslant 2$ the function $F: D^n\to\mathbb{R}$ given by*

$$F(x_1,\dots,x_n)=f(x_1)+\dots+f(x_n),\quad (x_1,\dots,x_n)\in D^n,$$

is strongly Schur–convex with modulus c.

Proof. Assume that $f : D \to \mathbb{R}$ is strongly convex with modulus c. Since X is an inner product space, the function $h : D \to \mathbb{R}$ given by $h(x) = f(x) - c\|x\|^2$, $x \in D$, is convex (see [10, Lemma 2.1]). Let $x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in D^n$ and $x \preccurlyeq y$. There exists a doubly stochastic $n \times n$ matrix $P = [t_{ij}]$ such that $x = y \cdot P$. Then

$$x_j = \sum_{i=1}^{n} t_{ij} y_i, \quad j = 1, \dots, n,$$

and, by the convexity of h, we obtain

$$\begin{aligned} h(x_1) + \dots + h(x_n) &= \sum_{j=1}^{n} h\Big(\sum_{i=1}^{n} t_{ij} y_i\Big) \leqslant \sum_{j=1}^{n}\sum_{i=1}^{n} t_{ij} h(y_i) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij} h(y_i) = \sum_{i=1}^{n} h(y_i) \sum_{j=1}^{n} t_{ij} = h(y_1) + \dots + h(y_n). \end{aligned}$$

Consequently,

$$\begin{aligned} F(x) &= f(x_1) + \dots + f(x_n) \\ &= h(x_1) + \dots + h(x_n) + c\big(\|x_1\|^2 + \dots + \|x_n\|^2\big) \\ &\leqslant h(y_1) + \dots + h(y_n) + c\big(\|x_1\|^2 + \dots + \|x_n\|^2\big) \\ &= f(y_1) + \dots + f(y_n) - c\big(\|y_1\|^2 + \dots + \|y_n\|^2\big) + c\big(\|x_1\|^2 + \dots + \|x_n\|^2\big) \\ &= F(y) - c\big(|\!|\!|y|\!|\!|^2 - |\!|\!|x|\!|\!|^2\big). \end{aligned}$$

This shows that F is strongly Schur–convex with modulus c, which was to be proved. $\square$

Remark 2. The converse theorem is not true. For instance, if $a : \mathbb{R} \to \mathbb{R}$ is an additive discontinuous function, then $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = a(x) + x^2$, $x \in \mathbb{R}$, is not strongly convex with any $c > 0$ (because it is not continuous) but it generates strongly Schur–convex sums. To see this take $x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in \mathbb{R}^n$ $(n \geqslant 2)$ such that $x \preccurlyeq y$. Then $x = y \cdot P$ for some doubly stochastic $n \times n$ matrix $P = [t_{ij}]$. By the additivity of a we have

$$\begin{aligned} a(x_1) + \dots + a(x_n) &= a(x_1 + \dots + x_n) = a\Big(\sum_{j=1}^{n}\sum_{i=1}^{n} t_{ij} y_i\Big) \\ &= a\Big(\sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij} y_i\Big) = a\Big(\sum_{i=1}^{n} y_i \sum_{j=1}^{n} t_{ij}\Big) = a(y_1) + \dots + a(y_n). \end{aligned}$$

Hence,

$$\begin{aligned} f(x_1) + \dots + f(x_n) &= a(x_1) + \dots + a(x_n) + x_1^2 + \dots + x_n^2 \\ &= a(y_1) + \dots + a(y_n) + y_1^2 + \dots + y_n^2 - (y_1^2 + \dots + y_n^2 - x_1^2 - \dots - x_n^2) \\ &= f(y_1) + \dots + f(y_n) - \big(\|y\|^2 - \|x\|^2\big). \end{aligned}$$

This proves that $F : \mathbb{R}^n \to \mathbb{R}$ defined by $F(x_1, \dots, x_n) = f(x_1) + \dots + f(x_n)$ is strongly Schur–convex with modulus 1.

The next result shows that the strong Jensen–convexity is a necessary condition under which f generates strongly Schur–convex sums.

Theorem 3. *Let D be a convex subset of an inner product space $\big(X,\|\cdot\|\big)$, $c>0$ and $f:D\to\mathbb{R}$. If for some $n\geqslant 2$ the function $F:D^n\to\mathbb{R}$ given by*

$$F(x_1,\dots,x_n)=f(x_1)+\dots+f(x_n),\quad (x_1,\dots,x_n)\in D^n$$

is strongly Schur–convex with modulus c, then f is strongly Jensen–convex with modulus c.

Proof. Take $y_1,y_2\in D$ and put $x_1=x_2=\frac{1}{2}(y_1+y_2)$. Consider the points

$$y=(y_1,y_2,y_2,\dots,y_2),\quad x=(x_1,x_2,y_2,\dots,y_2)$$

(if $n=2$, then we take $y=(y_1,y_2)$, $x=(x_1,x_2)$). Now, if

$$P=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&0&\cdots&0\\ \frac{1}{2}&\frac{1}{2}&0&\cdots&0\\ 0&0&1&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&\cdots&1\end{bmatrix}$$

then $x=y\cdot P$ and $x\preccurlyeq y$. Therefore, by the strong Schur–convexity of F,

$$F(x)\leqslant F(y)-c\big(|\!|\!|y|\!|\!|^2-|\!|\!|x|\!|\!|^2\big),$$

whence

$$2f\Big(\frac{y_1+y_2}{2}\Big)\leqslant f(y_1)+f(y_2)-c\Big(\|y_1\|^2+\|y_2\|^2-2\Big\|\frac{y_1+y_2}{2}\Big\|^2\Big). \tag{2}$$

By the parallelogram law we have

$$\|y_1\|^2+\|y_2\|^2=\frac{1}{2}\|y_1+y_2\|^2+\frac{1}{2}\|y_1-y_2\|^2.$$

Consequently, by (2),

$$f\Big(\frac{y_1+y_2}{2}\Big)\leqslant\frac{f(y_1)+f(y_2)}{2}-\frac{c}{4}\|y_1-y_2\|^2,$$

which means that f is strongly Jensen–convex with modulus c. □

Remark 4. The converse theorem is not true. For instance, let $a:\mathbb{R}\to\mathbb{R}$ be an additive discontinuous function such that $a(1)=0$ and let $t\in(0,1)$ with $a(t)\neq 0$. Then the function $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=\big|a(x)\big|+x^2$, $x\in\mathbb{R}$, is strongly Jensen–convex with modulus 1 (because $x\mapsto f(x)-x^2=\big|a(x)\big|$ is a Jensen–convex function, cf. [10, Lemma 2.1]), but it does not generate strongly Schur–convex sums with modulus 1. Indeed, if $n=2$, $x=(t,1-t)$ and $y=(1,0)$, then $x\preccurlyeq y$, but

$$F(x)=\big|a(t)\big|+\big|a(1-t)\big|+t^2+(1-t)^2>t^2+(1-t)^2=F(y)-\big(|\!|\!|y|\!|\!|^2-|\!|\!|x|\!|\!|^2\big).$$

Since continuous strongly Jensen–convex functions are strongly convex (see [1, Corollary 2.2]), as an immediate consequence of Theorem 3 we obtain the following result.

Corollary 5. *Let D be a convex subset of an inner product space $\big(X,\|\cdot\|\big)$ and $c>0$. If $f:D\to\mathbb{R}$ is continuous and for some $n\geqslant 2$ the function $F:D^n\to\mathbb{R}$ given by $F(x_1,\dots,x_n)=f(x_1)+\dots+f(x_n)$, $(x_1,\dots,x_n)\in D^n$, is strongly Schur–convex with modulus c, then f is strongly convex with modulus c.*

Remark 6. In the above Corollary the assumption that f is continuous can be replaced by other, formally weaker, regularity conditions. For instance, we may assume that f is bounded from above on a set with nonempty interior or f is Lebesgue measurable (if $X = \mathbb{R}^n$). It is a consequence of Bernstein–Doetsch or Sierpiński type results stating that under such assumptions strongly Jensen–convex functions are strongly convex (cf. [1]; for comprehensive review of theorems guaranteeing continuity of Jensen–convex functions see also [5]).

3. A CHARACTERIZATION

In this section we characterize the functions generating Schur–convex sums without any regularity assumptions. Our result is a counterpart of the celebrated Ng's Theorem [9] giving a representation of functions generating Schur–convex sums.

Theorem 7. *Let D be a convex subset of an inner product space $\big(X, \|\cdot\|\big)$, $f : D \to \mathbb{R}$ and $c > 0$. The following conditions are equivalent.*

(i) *For every $n \geqslant 2$ the function $F : D^n \to \mathbb{R}$ defined by*

$$F(x_1, \dots, x_n) = f(x_1) + \cdots + f(x_n), \quad (x_1, \dots, x_n) \in D^n, \tag{3}$$

is strongly Schur–convex with modulus c.

(ii) *For some $n \geqslant 2$ the function F given by* (3) *is strongly Schur–convex with modulus c.*

(iii) *The function f is strongly Wright–convex with modulus c.*

(iv) *There exist a convex function $g : D \to \mathbb{R}$ and an additive function $a : X \to \mathbb{R}$ such that*

$$f(x) = g(x) + a(x) + c\|x\|^2, \quad x \in D. \tag{4}$$

Proof. The implication (i) $\Longrightarrow$ (ii) is obvious.

To prove (ii) $\Longrightarrow$ (iii) fix $y_1, y_2 \in D$ and $t \in (0, 1)$. Put

$$x_1 = ty_1 + (1-t)y_2, \quad x_2 = (1-t)y_1 + ty_2$$

and, if $n > 2$, take additionally $x_i = y_i = z \in D$ for $i = 3, \dots, n$. Then, by the similar argumentation as in the proof of Theorem 3, we have

$$x = (x_1, \dots, x_n) \preccurlyeq y = (y_1, \dots, y_n).$$

Therefore, using the strong convexity of F, we obtain

$$F(x) \leqslant F(y) - c\big(|\!|\!|y|\!|\!|^2 - |\!|\!|x|\!|\!|^2\big),$$

and hence

$$\begin{aligned} f\big(ty_1 + (1-t)y_2\big) + f\big((1-t)y_1 + ty_2\big) \leqslant f(y_1) + f(y_2)& \\ - c\big(\|y_1\|^2 + \|y_2\|^2 - \|ty_1 + (1-t)y_2\|^2 - \|(1-t)y_1 + ty_2\|^2\big)&. \end{aligned} \tag{5}$$

Using elementary properties of the inner product we get

$$\begin{aligned} &\|y_1\|^2 + \|y_2\|^2 - \|ty_1 + (1-t)y_2\|^2 - \|(1-t)y_1 + ty_2\|^2 \\ &= \|y_1\|^2 + \|y_2\|^2 \\ &\quad - \big(t^2\|y_1\|^2 + (1-t)^2\|y_2\|^2 + (1-t)^2\|y_1\|^2 + t^2\|y_2\|^2 + 4t(1-t)\langle y_1|y_2\rangle\big) \\ &= 2t(1-t)\big(\|y_1\|^2 - 2\langle y_1|y_2\rangle + \|y_2\|^2\big) = 2t(1-t)\|y_1 - y_2\|^2. \end{aligned}$$

Consequently, from (5) we get

$$f\big(ty_1 + (1-t)y_2\big) + f\big((1-t)y_1 + ty_2\big) \leqslant f(y_1) + f(y_2) - 2ct(1-t)\|y_1 - y_2\|^2,$$

which means that f is strongly Wright–convex with modulus c.

The implication (iii) $\Longrightarrow$ (iv) follows from the characterization of strongly Wright–convex functions given in [8, Corollary 5].

To see that (iv) $\Longrightarrow$ (i) assume that f has the representation (4). Then the function $h = g + c\|\cdot\|^2$ is strongly convex with modulus c (cf. [10]) and hence, by Theorem 1, it generates strongly Schur–convex sums. Therefore, for any $x = (x_1, \dots, x_n) \preccurlyeq y = (y_1, \dots, y_n)$ we have

$$h(x_1) + \dots + h(x_n) \leqslant h(y_1) + \dots + h(y_n) - c\big(|||y|||^2 - |||x|||^2\big).$$

Consequently, using the additivity of a (similarly as in Remark 2), we arrive at

$$\begin{aligned} F(x) &= f(x_1) + \dots + f(x_n) = h(x_1) + \dots + h(x_n) + a(x_1) + \dots + a(x_n) \\ &\leqslant h(y_1) + \dots + h(y_n) - c\big(|||y|||^2 - |||x|||^2\big) + a(y_1) + \dots + a(y_n) \\ &= f(y_1) + \dots + f(y_n) - c\big(|||y|||^2 - |||x|||^2\big) = F(y) - c\big(|||y|||^2 - |||x|||^2\big), \end{aligned}$$

which shows that F is strongly Schur–convex with modulus c. This finishes the proof. $\square$

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