ON FUNCTIONAL EQUATIONS CONNECTED WITH QUADRATURE RULES

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Abstract. The functional equations of the form

$$F(y)-F(x)=(y-x)\Big[\alpha_0 f(x)+\sum_{k=1}^{n}\alpha_k f\big(\lambda_k x+(1-\lambda_k)y\big)+\alpha_{n+1}f(y)\Big]$$

are considered. They are connected with quadrature rules of the approximate integration. We show that such equations characterize polynomials in the class of continuous functions. It is also shown that if the number of components is sufficiently small, then the continuity is forced by the equation itself.

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1. Introduction

It is well–known that if $I\subset\mathbb{R}$ is an interval and the function $f:I\to\mathbb{R}$ is convex, then for all $x,y\in I$ the following Hermite–Hadamard inequality is satisfied:

$$f\Big(\frac{x+y}{2}\Big)\le\frac{1}{y-x}\int_x^y f(t)dt\le\frac{f(x)+f(y)}{2}\,. \tag{1}$$

The interesting study of this inequality and lots of related inequalities was given by Dragomir and Pearce [2]. Rewrite now the inequality (1) in the form

$$\frac{1}{y-x}\int_x^y f(t)dt\in\Big[f\Big(\frac{x+y}{2}\Big),\frac{f(x)+f(y)}{2}\Big].$$

However, if we consider the function $f(x)=x^2$, then we have much more detailed information. Namely,

$$\frac{1}{y-x}\int_x^y f(t)dt=\frac{2}{3}f\Big(\frac{x+y}{2}\Big)+\frac{1}{3}\frac{f(x)+f(y)}{2}\,.$$

Therefore in the paper [4] the equation

$$F(y)-F(x)=(y-x)\Big[\frac{k}{k+1}f\Big(\frac{x+y}{2}\Big)+\frac{1}{k+1}\frac{f(x)+f(y)}{2}\Big] \tag{2}$$

was considered. It was proved (for functions acting on integral domains divisible by 2) that if a pair F, f satisfies (2), then f must be a polynomial function of order not greater than 3.

In this paper by a *polynomial function of order* n we understand any solution of the functional equation $\Delta_h^{n+1} f(x) = 0$, where Δ_h^n stands for the n–th iterate of the difference operator $\Delta_h f(x) = f(x+h) - f(x)$. Observe that a continuous polynomial function of order n is a polynomial of degree at most n (see [6, Theorem 4, p. 398]).

One should also mention that the equation (2) (with $k = 1$) was considered in [1]. On the other hand, in the monograph [9], more general version of (2) was treated for functions acting on $\mathbb{R}$.

In this paper we would like to modify the equation (2) in order to characterize polynomials of higher degree. To this end let us observe that one could look at the right–hand side of (2) as the difference $(y - x)$ multiplied by a convex combination of the values $f(x)$, $f(y)$, $f(\frac{x+y}{2})$. Therefore we suppose that $\frac{F(y)-F(x)}{y-x}$ is a combination of more than three points of the form $f\big(\lambda x + (1-\lambda)y\big)$ and we arrive at the functional equation of the form

$$F(y) - F(x) = (y - x) \sum_{k=1}^{n} \alpha_k f\big(\lambda_k x + (1-\lambda_k)y\big), \tag{3}$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ (not necessarily in $[0, 1]$) and $\alpha_1, \ldots, \alpha_n$ are real numbers summing up to 1. This equation is related to the approximate integration. Namely, it is not difficult to observe that if f is a continuous solution of (3), then $F' = f$, whence (3) can be expressed in the form

$$\frac{1}{y - x} \int_x^y f(t)dt = \sum_{k=1}^{n} \alpha_k f\big(\lambda_k x + (1-\lambda_k)y\big). \tag{4}$$

The quadrature rules of an approximate integration can be obtained by the appropriate specification of the coefficients of (4). For that reason, in the study of the equation (3) we distinguish three cases. In the first one there are no endpoints x, y on the right–hand side of (3) (cf. the equation (16); observe that Gauss–Legendre quadrature formula contains only the interior nodes). In the second case we suppose that the right–hand side of (3) contains one endpoint x, which corresponds to the Radau quadrature (cf. the equation (19)). The third case concerns the situation, when the right–hand side of (3) contains both x and y, which is connected with the Lobatto quadrature (cf. the equation (20)).

Notice that the approach of numerical analysis is the following: assuming that equations of the form (4) hold for polynomials of possibly maximal degree (say N) determine the coefficients λ_k, α_k $(k = 1, \ldots, n)$ (see e.g. [8, 11]). The examples of polynomials of degree greater than N which do not fulfil such equations are well–known.

Our approach differs from the one described above. Namely, we prove in Theorems 1, 2 and 3, that if a continuous function f fulfils some equation of the form (4) (or, more generally, (3)), then it must be a polynomial of degree not greater than some positive integer. Thus we answer the converse question to that of the numerical analysis.

2. Preliminaries

We obtain our results using the lemma established by M. Sablik [10, Lemma 2.3] and improved by I. Pawlikowska [7]. To quote it we need some notations. Let G, H be the Abelian groups and $SA^0(G,H) := H$, $SA^1(G,H) := \mathrm{Hom}(G,H)$ (i.e. the group of all homomorphisms from G into H), and for $i \in \mathbb{N}$, $i \geq 2$, let $SA^i(G,H)$ be the group of all i–additive and symmetric mappings from G^i into H. Furthermore, let $\mathscr{P} := \big\{(\alpha,\beta) \in \mathrm{Hom}(G,G)^2 : \alpha(G) \subset \beta(G)\big\}$. Finally, for $x \in G$, let $x^i = \underbrace{(x,\ldots,x)}_{i}$, $i \in \mathbb{N}$.

Lemma 1. *Fix $N \in \mathbb{N} \cup \{0\}$ and let $I_0,\ldots,I_N$ be finite subsets of $\mathscr{P}$. Suppose that H is uniquely divisible by $N!$ and let the functions $\varphi_i : G \to SA^i(G,H)$ and $\psi_{i,(\alpha,\beta)} : G \to SA^i(G,H)$ ($(\alpha,\beta) \in I_i$, $i = 0,\ldots,N$) satisfy*

$$\varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \varphi_i(x)(y^i) = \sum_{i=0}^{N} \sum_{(\alpha,\beta)\in I_i} \psi_{i,(\alpha,\beta)}\big(\alpha(x)+\beta(y)\big)(y^i)$$

for every $x, y \in G$. Then φ_N is a polynomial function of order at most $k-1$, where

$$k = \sum_{i=0}^{N} \mathrm{card}\Big(\bigcup_{s=i}^{N} I_s\Big).$$

In the original Sablik's result the order of φ_N was not greater than k, while Pawlikowska noticed that it can be lowered by 1.

In our considerations we use Lemma 1 only for $G = H = \mathbb{R}$, $N = 1$ and we consider only homomorphisms of the form $x \mapsto \alpha x$, $\alpha \in \mathbb{R}$. In this setting Lemma 1 can be rephrased in the form

Lemma 2. *Let I_0, I_1 be finite subsets of $\mathscr{P}$, where*

$$\mathscr{P} = \big\{(\alpha,\beta) \in \mathbb{R}^2 : \alpha = 0 \text{ or } \beta \neq 0\big\}.$$

Let the functions $\varphi_i, \psi_{i,(\alpha,\beta)} : \mathbb{R} \to \mathbb{R}$ ($(\alpha,\beta) \in I_i$, $i = 0,1$) satisfy

$$\varphi_1(x)y + \varphi_0(x) = \sum_{(\alpha,\beta)\in I_0} \psi_{0,(\alpha,\beta)}(\alpha x + \beta y) + y\Big[\sum_{(\alpha,\beta)\in I_1} \psi_{1,(\alpha,\beta)}(\alpha x + \beta y)\Big] \tag{5}$$

for any $x, y \in \mathbb{R}$. Then φ_1 is a polynomial function of order at most k, where

$$k = \mathrm{card}(I_0 \cup I_1) + \mathrm{card}\, I_1 - 1.$$

In the Section 3.5 of the monograph [9] (Theorems 3.10 and 3.11) the general solution of the functional equation

$$F(y) - G(x) = (y-x)\big[f(x) + g(sx+ty) + h(y)\big]$$

is given for functions mapping $\mathbb{R}$ into $\mathbb{R}$. The lemma below generalizes "if" parts of these results.

Lemma 3. *Let $n \in \mathbb{N}$. Assume that*

(i) $s_1, \dots, s_n, t_1, \dots, t_n \in \mathbb{R} \setminus \{0\}$,
(ii) $s_k + t_k \neq 0,\ k = 1, \dots, n$,
(iii) $\left|\begin{smallmatrix} s_k & t_k \\ s_l & t_l \end{smallmatrix}\right| \neq 0,\ k \neq l,\ k, l \in \{1, \dots, n\}$,
(iv) *the functions* $F, f_0, f_1, \dots, f_n, f_{n+1} : \mathbb{R} \to \mathbb{R}$ *fulfil for any* $x, y \in \mathbb{R}$ *the functional equation*

$$F(y) - F(x) = (y - x)\Big[f_0(x) + \sum_{k=1}^{n} f_k(s_k x + t_k y) + f_{n+1}(y)\Big]. \tag{6}$$

Then

(a) *If* $f_0 = f_{n+1} \equiv 0$, *then* $f_1, \dots, f_n$ *are the polynomial functions of order at most* $2n - 1$.
(b) *If* $f_0 \not\equiv 0$, $f_{n+1} \equiv 0$ *or* $f_0 \equiv 0$, $f_{n+1} \not\equiv 0$, *then* $f_0, f_1, \dots, f_n, f_{n+1}$ *are the polynomial functions of order at most* $2n$.
(c) *If* $f_0 \not\equiv 0$, $f_{n+1} \not\equiv 0$, *then* $f_0, f_1, \dots, f_n, f_{n+1}$ *are the polynomial functions of order at most* $2n + 1$.

Proof. Let $s_0 = 1,\ t_0 = 0;\ s_{n+1} = 0,\ t_{n+1} = 1$. Fix $m \in \{0, \dots, n + 1\}$ and interchange x, y in (6) with $\frac{x - t_m y}{s_m + t_m}$, $\frac{x + s_m y}{s_m + t_m}$, respectively. Performing elementary computations we obtain

$$y f_m(x) = F\Big(\frac{x + s_m y}{s_m + t_m}\Big) - F\Big(\frac{x - t_m y}{s_m + t_m}\Big) - y \sum_{\substack{k=0 \\ k \neq m}}^{n+1} f_k\Big(\frac{s_k + t_k}{s_m + t_m} x + \frac{s_m t_k - s_k t_m}{s_m + t_m} y\Big).$$

Denote

$$\alpha_k = \frac{s_k + t_k}{s_m + t_m}, \quad \beta_k = \frac{\left|\begin{smallmatrix} s_m & t_m \\ s_k & t_k \end{smallmatrix}\right|}{s_m + t_m}, \qquad k = 0, \dots, n + 1.$$

Then

$$y f_m(x) = F(\alpha_{n+1} x + \beta_{n+1} y) - F(\alpha_0 x + \beta_0 y) - y \sum_{\substack{k=0 \\ k \neq m}}^{n+1} f_k(\alpha_k x + \beta_k y). \tag{7}$$

We have

$$\alpha_m = 1, \quad \beta_m = 0. \tag{8}$$

Observe that if $k = 0, \dots, n + 1,\ k \neq m$, then by (iii) we have $\beta_k \neq 0$, whence

$$(\alpha_k, \beta_k) \in \mathscr{P}, \text{ whenever } k = 0, \dots, n + 1,\ k \neq m, \tag{9}$$

where $\mathscr{P}$ is a set defined in Lemma 2. Observe also that

$$(\alpha_k, \beta_k) \neq (\alpha_l, \beta_l), \text{ whenever } k \neq l,\ k, l \in \{0, \dots, n + 1\}. \tag{10}$$

Indeed, if $k \neq m$, then $\beta_k \neq 0 = \beta_m$ by (iii) and (8). If $(\alpha_k, \beta_k) = (\alpha_l, \beta_l)$ for some $k \neq l,\ k, l \neq m$, then

$$s_k + t_k = s_l + t_l \qquad \text{and} \qquad \left|\begin{smallmatrix} s_m & t_m \\ s_k & t_k \end{smallmatrix}\right| = \left|\begin{smallmatrix} s_m & t_m \\ s_l & t_l \end{smallmatrix}\right|,$$

whence $(s_k - s_l)(s_m + t_m) = 0$, which together with (ii) gives $s_k = s_l,\ t_k = t_l$. This contradicts (iii).

Now we will prove (a). Assume $f_0 = f_{n+1} \equiv 0$. Let $m \in \{1, \dots, n\}$. By (7)

$$y f_m(x) = F(\alpha_{n+1}x + \beta_{n+1}y) - F(\alpha_0 x + \beta_0 y) - y \sum_{\substack{k=1 \\ k \neq m}}^{n} f_k(\alpha_k x + \beta_k y). \tag{11}$$

Let

$$I_0 = \big\{(\alpha_0, \beta_0), (\alpha_{n+1}, \beta_{n+1})\big\}, \quad I_1 = \big\{(\alpha_k, \beta_k) : k \neq m,\ k = 1, \dots, n\big\}.$$

By (9), $I_0, I_1 \subset \mathscr{P}$, and by (10), $\operatorname{card} I_0 = 2$, $\operatorname{card} I_1 = n - 1$, $I_0 \cap I_1 = \varnothing$. Denote

$$\begin{aligned} &\varphi_0 \equiv 0, \quad \varphi_1 = f_m, \quad \psi_{0,(\alpha_0,\beta_0)} = -F, \quad \psi_{0,(\alpha_{n+1},\beta_{n+1})} = F \\ \text{and} \quad &\psi_{1,(\alpha_k,\beta_k)} = -f_k \quad \text{for } k \neq m,\ k = 1, \dots, n. \end{aligned}$$

Thus (11) has the form (5). By Lemma 2 we infer that f_m is a polynomial function of order at most $\operatorname{card}(I_0 \cup I_1) + \operatorname{card} I_1 - 1 = 2n - 1$ and (a) is proved.

To prove (b) assume $f_0 \not\equiv 0$, $f_{n+1} \equiv 0$. Then for $m \in \{0, \dots, n\}$, (7) yields

$$y f_m(x) = F(\alpha_{n+1}x + \beta_{n+1}y) - F(\alpha_0 x + \beta_0 y) - y \sum_{\substack{k=0 \\ k \neq m}}^{n} f_k(\alpha_k x + \beta_k y). \tag{12}$$

Assume now that $m \in \{1, \dots, n\}$ and let

$$\begin{aligned} I_0 &= \big\{(\alpha_0, \beta_0), (\alpha_{n+1}, \beta_{n+1})\big\}, \\ I_1 &= \big\{(\alpha_k, \beta_k) : k \neq m,\ k = 1, \dots, n\big\} \cup \big\{(\alpha_0, \beta_0)\big\}. \end{aligned}$$

By (9), $I_0, I_1 \subset \mathscr{P}$ and by (10), $\operatorname{card} I_0 = 2$, $\operatorname{card} I_1 = n$, $I_0 \cap I_1 = \big\{(\alpha_0, \beta_0)\big\}$. Denote

$$\begin{aligned} &\varphi_0 \equiv 0, \quad \varphi_1 = f_m, \quad \psi_{0,(\alpha_0,\beta_0)} = -F, \quad \psi_{0,(\alpha_{n+1},\beta_{n+1})} = F \\ \text{and} \quad &\psi_{1,(\alpha_k,\beta_k)} = -f_k \quad \text{for } k \neq m,\ k = 0, \dots, n. \end{aligned}$$

Thus (12) has the form (5). By Lemma 2 we infer that f_m is a polynomial function of order at most $\operatorname{card}(I_0 \cup I_1) + \operatorname{card} I_1 - 1 = 2n$.

Now let $m = 0$. Then rewrite (12) in the form

$$y f_0(x) + F(x) = F(\alpha_{n+1}x + \beta_{n+1}y) - y \sum_{k=1}^{n} f_k(\alpha_k x + \beta_k y) \tag{13}$$

(notice that by (8) for $m = 0$ we get $\alpha_0 = 1$, $\beta_0 = 0$). Let

$$I_0 = \big\{(\alpha_{n+1}, \beta_{n+1})\big\}, \quad I_1 = \big\{(\alpha_k, \beta_k) : k = 1, \dots, n\big\}.$$

By (9), $I_0, I_1 \subset \mathscr{P}$ and by (10), $\operatorname{card} I_1 = n$, $I_0 \cap I_1 = \varnothing$. Denote

$$\begin{aligned} &\varphi_0 = F, \quad \varphi_1 = f_0, \quad \psi_{0,(\alpha_{n+1},\beta_{n+1})} = F \\ \text{and} \quad &\psi_{1,(\alpha_k,\beta_k)} = -f_k \quad \text{for } k = 1, \dots, n. \end{aligned}$$

Thus (13) has the form (5). By Lemma 2 we infer that f_0 is a polynomial function of order at most $\operatorname{card}(I_0 \cup I_1) + \operatorname{card} I_1 - 1 = 2n$.

If $f_0 \equiv 0$, $f_{n+1} \not\equiv 0$, we proceed similarly and (b) is proved.

To show that (c) holds assume $f_0 \not\equiv 0$, $f_{n+1} \not\equiv 0$. For $m \in \{1,\dots,n\}$ rewrite (7) as follows:

$$
\begin{aligned}
(14)\quad yf_m(x) &= F(\alpha_{n+1}x+\beta_{n+1}y) - F(\alpha_0 x+\beta_0 y)\\
&\quad - y\Big[f_0(\alpha_0 x+\beta_0 y) + \sum_{\substack{k=1\\k\neq m}}^{n} f_k(\alpha_k x+\beta_k y) + f_{n+1}(\alpha_{n+1}x+\beta_{n+1}y)\Big].
\end{aligned}
$$

Let

$$
I_0=\big\{(\alpha_0,\beta_0),(\alpha_{n+1},\beta_{n+1})\big\},\quad I_1=\big\{(\alpha_k,\beta_k): k\neq m,\ k=1,\dots,n\big\}\cup I_0.
$$

By (9), $I_0, I_1 \subset \mathscr{P}$ and by (10), $\operatorname{card} I_0 = 2$, $\operatorname{card} I_1 = n+1$. Denote

$$
\begin{aligned}
&\varphi_0\equiv 0,\quad \varphi_1=f_m,\quad \psi_{0,(\alpha_0,\beta_0)}=-F,\quad \psi_{0,(\alpha_{n+1},\beta_{n+1})}=F\\
\text{and}\quad &\psi_{1,(\alpha_k,\beta_k)}=-f_k\quad \text{for } k\neq m,\ k=0,\dots,n+1.
\end{aligned}
$$

Thus (14) has the form (5). By Lemma 2 we infer that f_m is a polynomial function of order at most $\operatorname{card}(I_0\cup I_1)+\operatorname{card} I_1 - 1 = 2n+1$.

Now let $m=0$. Then rewriting (14) and using (8) we obtain

$$
\begin{aligned}
(15)\quad yf_0(x)+F(x) &= F(\alpha_{n+1}x+\beta_{n+1}y)\\
&\quad - y\Big[\sum_{k=1}^{n} f_k(\alpha_k x+\beta_k y) + f_{n+1}(\alpha_{n+1}x+\beta_{n+1}y)\Big].
\end{aligned}
$$

Let

$$
I_0=\big\{(\alpha_{n+1},\beta_{n+1})\big\},\quad I_1=\big\{(\alpha_k,\beta_k): k=1,\dots,n\big\}\cup I_0.
$$

Then by (9), $I_0, I_1 \subset \mathscr{P}$ and by (10), $\operatorname{card} I_1 = n+1$. Denote

$$
\begin{aligned}
&\varphi_0=F,\quad \varphi_1=f_0,\quad \psi_{0,(\alpha_{n+1},\beta_{n+1})}=F\\
\text{and}\quad &\psi_{1,(\alpha_k,\beta_k)}=-f_k\quad \text{for } k=1,\dots,n+1.
\end{aligned}
$$

Thus (15) has the form (5). By Lemma 2 we infer that f_0 is a polynomial function of order at most $\operatorname{card}(I_0\cup I_1)+\operatorname{card} I_1 - 1 = 2n+1$.

If $m=n+1$, then we proceed similarly and the proof of (c) is finished. $\square$

3. Results for continuous functions

We start with the functional equation connected with Gauss quadratures.

Theorem 1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function, $F:\mathbb{R}\to\mathbb{R}$ and $n\in\mathbb{N}$. The following conditions are equivalent:*

(i) *There exist distinct real numbers $\lambda_1,\dots,\lambda_n \notin \{0,1\}$ and $\alpha_1,\dots,\alpha_n\in\mathbb{R}$ with $\alpha_1+\dots+\alpha_n=1$ such that F, f fulfil for any $x,y\in\mathbb{R}$ the functional equation*

$$
F(y)-F(x)=(y-x)\sum_{k=1}^{n}\alpha_k f\big(\lambda_k x+(1-\lambda_k)y\big). \tag{16}
$$

(ii) *The function f is a polynomial of degree at most $2n-1$ and F is the primitive function for f.*

Proof. To prove (i)$\Rightarrow$(ii) observe that $f_k = \alpha_k f$, $s_k = \lambda_k$, $t_k = 1-\lambda_k$ ($k = 1,\dots,n$) fulfil the assumptions of Lemma 3. Then by the assertion (a), f is a polynomial function of order at most $2n-1$ and by continuity f is a polynomial of degree at most $2n-1$ (cf. Remark 1 at the end of this section). Dividing both sides of (16) by $y-x \neq 0$ and tending with y to x we obtain $F'(x) = (\alpha_1+\dots+\alpha_n)f(x) = f(x)$.

To prove the converse implication consider Gauss–Legendre quadrature on $[-1,1]$. It is well–known from numerical analysis that there exist positive numbers $\alpha_1,\dots,\alpha_n$ summing up to 1 and distinct $t_1,\dots,t_n \in (-1,1)$ such that the equation

$$\frac{1}{2}\int_{-1}^{1} g(t)dt = \sum_{k=1}^{n} \alpha_k g(t_k) \tag{17}$$

holds for all polynomials of degree at most $2n-1$. For the exact values of the nodes $t_1,\dots,t_n$ and the weights $\alpha_1,\dots,\alpha_n$ see e.g. [8, 12]. Now let f be a polynomial of degree at most $2n-1$. Fix $x,y\in\mathbb{R}$, $x\neq y$. Since $F' = f$, we have

$$F(y)-F(x) = \int_{x}^{y} f(u)du. \tag{18}$$

For $z\in[-1,1]$ let $u = \frac{1-z}{2}x+\frac{1+z}{2}y$. If $g(z) = f(u)$, then g is also the polynomial of degree at most $2n-1$. By (17), (18) and the above substitution

$$\begin{aligned}\frac{F(y)-F(x)}{y-x} &= \frac{1}{y-x}\int_{x}^{y} f(u)du = \frac{1}{2}\int_{-1}^{1} g(t)dt = \sum_{k=1}^{n}\alpha_k g(t_k)\\ &= \sum_{k=1}^{n}\alpha_k f\Big(\frac{1-t_k}{2}x+\frac{1+t_k}{2}y\Big).\end{aligned}$$

Multiplying both sides of this equation by $y-x$ we obtain (16) for $\lambda_k = \frac{1-t_k}{2}$, $k=1,\dots,n$. □

Our next result concerns the functional equation connected with Radau quadratures.

Theorem 2. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function, $F:\mathbb{R}\to\mathbb{R}$ and $n\in\mathbb{N}$. The following conditions are equivalent:*

(i) *There exist distinct real numbers $\lambda_1,\dots,\lambda_n \notin \{0,1\}$ and $\alpha_0,\dots,\alpha_n\in\mathbb{R}$ with $\sum_{k=0}^{n}\alpha_k = 1$ such that F, f fulfil for any $x,y\in\mathbb{R}$ the functional equation*

$$F(y)-F(x) = (y-x)\Big[\alpha_0 f(x) + \sum_{k=1}^{n}\alpha_k f\big((1-\lambda_k)x+\lambda_k y\big)\Big]. \tag{19}$$

(ii) *The function f is a polynomial of degree at most $2n$ and F is the primitive function for f.*

Proof. We proceed similarly as in the proof of Theorem 1. To prove (i)$\Rightarrow$(ii) we use the assertion (b) of Lemma 3. For the proof of the converse implication we consider Radau quadrature on $[-1,1]$. It is well–known from numerical analysis that there exist positive numbers $\alpha_0,\dots,\alpha_n$ summing up to 1 and distinct $t_1,\dots,t_n\in(-1,1)$ such that the equation

$$\frac{1}{2}\int_{-1}^{1} g(t)dt=\alpha_0 g(-1)+\sum_{k=1}^{n}\alpha_k g(t_k)$$

holds for all polynomials of degree at most $2n$. For the exact values of the nodes $t_1,\dots,t_n$ and the weights $\alpha_0,\dots,\alpha_n$ see e.g. [8, 14]. □

Now we deal with the functional equation connected with Lobatto quadratures.

Theorem 3. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function, $F:\mathbb{R}\to\mathbb{R}$ and $n\in\mathbb{N}$. The following conditions are equivalent:*

(i) *There exist distinct real numbers $\lambda_1,\dots,\lambda_n\notin\{0,1\}$ and $\alpha_0,\dots,\alpha_{n+1}\in\mathbb{R}$ with $\sum_{k=0}^{n}\alpha_k=1$ such that F,f fulfil for any $x,y\in\mathbb{R}$ the functional equation*

$$F(y)-F(x)=(y-x)\Big[\alpha_0 f(x)+\sum_{k=1}^{n}\alpha_k f\big((1-\lambda_k)x+\lambda_k y\big)+\alpha_{n+1}f(y)\Big]. \tag{20}$$

(ii) *The function f is a polynomial of degree at most $2n+1$ and F is the primitive function for f.*

Proof. We also proceed similarly as in the proof of Theorem 1. To prove (i)$\Rightarrow$(ii) we use the assertion (c) of Lemma 3. To prove the converse implication we consider Lobatto quadrature on $[-1,1]$. It is well–known from numerical analysis that there exist positive numbers $\alpha_0,\dots,\alpha_{n+1}$ summing up to 1 and distinct $t_1,\dots,t_n\in(-1,1)$ such that the equation

$$\frac{1}{2}\int_{-1}^{1} g(t)dt=\alpha_0 g(-1)+\sum_{k=1}^{n}\alpha_k g(t_k)+\alpha_{n+1}g(1)$$

holds for all polynomials of degree at most $2n+1$. For the exact values of the nodes $t_1,\dots,t_n$ and the weights $\alpha_0,\dots,\alpha_{n+1}$ see e.g. [8, 13]. □

Remark 1. Polynomial functions need not in general to be continuous. However, slight regularity assumptions force continuity. Thus, following Kuczma's book [6, pp. 380–386], the continuity assumption in Theorems 1, 2 and 3 can be replaced by one of listed below.

1. Boundedness on the set with nonempty interior.
2. Boundedness on the set of positive Lebesgue measure.
3. Boundedness on the second category set with the Baire property.
4. Local boundedness at one point.
5. Measurability.

4. Results without continuity

For the equations of the form (16), (19), it turns out that if n is sufficiently small, then continuity is forced by the equation itself. First we consider (19) with $n = 1$.

Theorem 4. *Let $\alpha \in \mathbb{R} \setminus \{0\}$ and let $\lambda \notin \{0, 1\}$ be a rational number. If $F, f : \mathbb{R} \to \mathbb{R}$ fulfil for any $x, y \in \mathbb{R}$ the functional equation*

$$F(y) - F(x) = (y - x)\Big[\alpha f(x) + (1 - \alpha) f\big(\lambda x + (1 - \lambda) y\big)\Big], \tag{21}$$

then

$$f(x) = ax^2 + bx + c \quad \text{and} \quad F(x) = \tfrac{1}{3}ax^3 + \tfrac{1}{2}bx^2 + cx + d$$

for some constants $a, b, c, d \in \mathbb{R}$.

Conversely, the functions f, F given by the above formulae, satisfy (21) with $\alpha = \frac{1}{4}$ and $\lambda = \frac{1}{3}$.

Proof. In Lemma 3 take $n = 1$. Observe that $s_1 = \lambda$, $t_1 = 1 - \lambda$, fulfil the assumptions (i), (ii) and (iii). Let $f_1 = \alpha f$, $f_2 = (1 - \alpha) f$. The assertion (b) implies that f is a polynomial function of order at most 2. It is well–known (see e.g. [6, Theorem 2, p. 395]) that

$$f(x) = a_2(x) + a_1(x) + c,$$

where $c \in \mathbb{R}$ is some constant and a_k is a diagonalization of some symmetric k–additive function $A_k : \mathbb{R}^k \to \mathbb{R}$ $(k = 1, 2)$. It is proved in [5, Lemma 3] that a_1, a_2 fulfil (21) with appropriately choosen function F. Denote

$$\begin{aligned}
\eta &:= (1 - \alpha)\lambda(1 - \lambda),\\
\gamma &:= \alpha + (1 - \alpha)\lambda^2,\\
\delta &:= (1 - \alpha)(1 - \lambda)^2.
\end{aligned}$$

It is proved in [5, Lemma 5] that if the condition $\eta = \gamma = \delta = 0$ does not hold (it is the case, since $\gamma + 2\eta\delta = 1$), then $a_2(x) = ax^2$ for some $a \in \mathbb{R}$. Now let

$$\gamma := \alpha + (1 - \alpha)\lambda, \qquad \delta := (1 - \alpha)(1 - \lambda).$$

It is proved in [5, Lemma 6] that if the condition $\gamma = \delta = 0$ does not hold (it is the case, since $\gamma + \delta = 1$), then $a_1(x) = bx$ for some $b \in \mathbb{R}$. We have proved that $f(x) = ax^2 + bx + c$. Now from the equation (21) we obtain

$$\frac{F(y) - F(x)}{y - x} = \alpha f(x) + (1 - \alpha) f(\lambda x + (1 - \lambda) y), \; x \neq y,$$

which, together with the form of f, means that

$$F(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + d$$

for some $d \in \mathbb{R}$. □

Now we shall deal with the equation (16) with $n = 2$, $\alpha_1 = \alpha_2 = \frac{1}{2}$ and $\lambda_2 = 1 - \lambda_1$. Using the result of the recent paper [3], one can easily prove the following

Theorem 5. *Let $\alpha \neq 0$, $\lambda \notin \{0,1\}$. If $F, f : \mathbb{R} \to \mathbb{R}$ fulfil for any $x, y \in \mathbb{R}$ the functional equation*

$$F(y) - F(x) = (y-x)\Big[\alpha f\big(\lambda x + (1-\lambda)y\big) + (1-\alpha) f\big((1-\lambda)x + \lambda y\big)\Big], \tag{22}$$

then

$$f(x) = ax^3 + bx^2 + cx + d,$$

$$F(x) = \tfrac{1}{4}ax^4 + \tfrac{1}{3}bx^3 + \tfrac{1}{2}cx^2 + dx + e$$

for some constants $a, b, c, d, e \in \mathbb{R}$.

Conversely, functions f, F given by the above formulas satisfy (22) *with $\alpha = \frac{1}{2}$ and $\lambda = \frac{3-\sqrt{3}}{6}$.*

Proof. First we assume that $\alpha = \frac{1}{2}$. Then we may use a result from [3, Theorem 2, part (i)] and we obtain that if $\lambda = \frac{1}{2}$, then $f(x) = ax + d$ for some $a, d \in \mathbb{R}$.

Now consider the case $\lambda \notin \left\{\frac{1}{2}, \frac{3-\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6}\right\}$. Then

$$\lambda^2 + (1-\lambda)^2 \neq 4\lambda(1-\lambda),$$

which, in view of [3, Theorem 2,part (v)], means that $f(x) = ax + d$ for some $a, d \in \mathbb{R}$.

Finally we consider the case $\lambda \in \left\{\frac{3-\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6}\right\}$. Then

$$\lambda^2 + (1-\lambda)^2 = 4\lambda(1-\lambda),$$

i.e. again from [3, Theorem 2,part (vi)] we obtain that $f(x) = cx^3 + bx^2 + ax + d$ for some constants $c, b, a, d \in \mathbb{R}$.

Now we will prove the assertion concerning the case $\alpha \neq \frac{1}{2}$. To this end, in (22) put $\frac{1}{\lambda}x$ in place of x and $\frac{1}{\lambda}y$ instead of y. Thus we have

$$F\big(\tfrac{1}{\lambda}y\big) - F\big(\tfrac{1}{\lambda}x\big) = \tfrac{1}{\lambda}(y-x)\Big[\alpha f\big(x + \tfrac{1-\lambda}{\lambda}y\big) + (1-\alpha) f\big(\tfrac{1-\lambda}{\lambda}x + y\big)\Big].$$

Multiplying this equation by λ and taking $G(x) := \lambda F\left(\frac{1}{\lambda}x\right)$, $\gamma := \frac{1-\lambda}{\lambda}$ we obtain that f satisfies

$$G(y) - G(x) = (y-x)[\alpha f(x + \gamma y) + (1-\alpha) f(\gamma x + y)]. \tag{23}$$

Interchanging here x with y we arrive at

$$G(y) - G(x) = (y-x)[\alpha f(\gamma x + y) + (1-\alpha) f(x + \gamma y)]. \tag{24}$$

Comparing equations (23) and (24), we obtain

$$\alpha f(x + \gamma y) + (1-\alpha) f(\gamma x + y) = \alpha f(\gamma x + y) + (1-\alpha) f(x + \gamma y)$$

and after some simple computations we get

$$(2\alpha - 1) f(x + \gamma y) = (2\alpha - 1) f(\gamma x + y).$$

But since $\alpha \neq \frac{1}{2}$, we have

$$f(x + \gamma y) = f(\gamma x + y).$$

Using this equality, we may rewrite (23) in the form

$$G(y) - G(x) = (y - x)[f(x + \gamma y)].$$

Now, using [9, Theorem 2.5, p. 44]), we infer that $f(x) = ax + d$ for some constants $a, d \in \mathbb{R}$.

By putting $x = 0$ in (22) we obtain the desired formula for F. Thus the proof is finished. □

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