Haar spaces and polynomial selections

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Abstract

A theorem on the existence of selections belonging to a Haar space is proved. As consequences some generalizations of the second author's earlier results concerning polynomial selections and separation by polynomials (cf. [7]) are obtained.

1. Introduction. Recall that if X, Y are two sets, then a map $\Phi : X \longrightarrow 2^Y$ is called a *multifunction* (or *set–valued function*). A function $h : X \longrightarrow Y$ is a *selection* of Φ if $h(x) \in \Phi(x)$ for every $x \in X$. For a topological vector space Y by $\mathrm{cc}(Y)$ we denote the family of all nonempty, compact and convex subsets of Y.

In [7] a necessary and sufficient condition for two real functions defined on a real interval I to be separated by a polynomial of degree at most n was given. It was obtained using a consequence of the classical Helly's theorem proved in [2] by E. Behrends and K. Nikodem. Their result was recently generalized by M. Balaj and K. Nikodem ([1, Theorem 1]) in the following way:

Theorem A. *Let D be a nonempty subset of a set X, $\big(Y, \|\cdot\|\big)$ be a normed space and for each $i \in \{1, \dots, l+1\}$ $\Phi_i : D \longrightarrow \mathrm{cc}(Y)$ be a multifunction. Assume that $\mathcal{W}$ is an l-dimensional subspace of the vector space of all functions from X to Y and D has enough points for $f|_D = 0$ to imply $f = 0$, for each $f \in \mathcal{W}$. If for every $l+1$ points $x_1, \dots, x_{l+1} \in D$ there exists an $h \in \mathcal{W}$ such that $h(x_i) \in \Phi_i(x_i)$, $(1 \leqslant i \leqslant l+1)$, then for some $i_0 \in \{1, \dots, l+1\}$ there exists an $\overline{h} \in \mathcal{W}$ such that $\overline{h}(x) \in \Phi_{i_0}(x)$ for all $x \in D$.*

The main goals of this note is to show that Theorem 1 of [7] can be generalized using Theorem A instead of Behrends and Nikodem's result and to give an extended version of Theorem 2 of [7] on separation by polynomials.

2. Haar spaces. Let us adopt the following definition (cf. [3], [4], [6]):

Definition. Let D be a set containing at least n elements. A linear subspace $\mathcal{H}_n(D)$ of $\mathbb{R}^D$ will be called an *n–dimensional Haar space on D*, if for any n distinct elements $x_1, x_2, \dots, x_n$ of D and any $y_1, y_2, \dots, y_n \in \mathbb{R}$ there is exactly one $h \in \mathcal{H}_n(D)$ for which $h(x_j) = y_j$ $(1 \leqslant j \leqslant n)$.

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By the above definition it follows immediately that any non–zero function $h \in \mathcal{H}_n(D)$ has at most $n-1$ zeros.

Borwein [3] gave some examples of Haar spaces as follows:

(i) $\operatorname{span}\{1, e^{\alpha_1 x}, \ldots, e^{\alpha_{n-1} x}\}$, where $\alpha_1, \ldots, \alpha_{n-1}$ are distinct non–zero real numbers, is an n–dimensional Haar space on any real interval;

(ii) $\operatorname{span}\big\{1, x, x^2, \ldots, x^{n-2}, f(x)\big\}$, where $f^{(n-1)}(x) > 0$, is an n–dimensional Haar space on any real interval;

(iii) $\operatorname{span}\{1, x^2, x^4, \ldots, x^{2n-2}\}$ is an n–dimensional Haar space on each interval $[a, b]$, where $a > 0$.

If $x_1, x_2, \ldots, x_n$ are n distinct elements of D, then for each $j \in \{1, 2, \ldots, n\}$ let $c_j(\,\cdot\,; x_1, x_2, \ldots, x_n)$ be the unique function in $\mathcal{H}_n(D)$ satisfying

$$c_j(x_i; x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \qquad (1 \leqslant i \leqslant n).$$

The proof of the following lemma is elementary.

Lemma 1. *Let $x_1, x_2, \ldots, x_n$ be distinct elements of D and $y_1, y_2, \ldots, y_n \in \mathbb{R}$. The unique function $h \in \mathcal{H}_n(D)$, for which $h(x_j) = y_j$ $(1 \leqslant j \leqslant n)$, is given by the formula*

$$h(x) = \sum_{j=1}^{n} c_j(x; x_1, x_2, \ldots, x_n) y_j.$$

As a consequence of Lemma 1 we get

$$\mathcal{H}_n(D) = \operatorname{span}\big\{c_j(\,\cdot\,; x_1, x_2, \ldots, x_n) \ : \ 1 \leqslant j \leqslant n\big\}$$

for every distinct points $x_1, \ldots, x_n \in D$. In particular, $\mathcal{H}_n(D)$ is n–dimensional space of functions.

3. Main result. Our main result is the following

Theorem 1. *Let $\mathcal{H}_n(D)$ be an n–dimensional Haar space and $\Phi_i : D \longrightarrow \operatorname{cc}(\mathbb{R})$ $(1 \leqslant i \leqslant n+1)$ be set–valued functions. Assume that:*

(i) for each $x \in D$, $\displaystyle\bigcap_{i=1}^{n+1} \Phi_i(x) \neq \emptyset$,

(ii) for any $n+1$ distinct elements $x_1, x_2, \ldots, x_{n+1} \in D$ there exists an index $i \in \{1, 2, \ldots, n+1\}$ such that

$$\Phi_i(x_i) \cap \sum_{\substack{j=1 \\ j \neq i}}^{n+1} c_j(x_i; x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) \Phi_j(x_j) \neq \emptyset.$$

Then there exist $i_0 \in \{1, 2, \dots, n+1\}$ and $h \in \mathcal{H}_n(D)$ such that $h(x) \in \Phi_{i_0}(x)$ for all $x \in D$.

Proof. On account of Theorem A it is enough to prove that for any $n+1$ points $x_1, x_2, \dots, x_{n+1} \in D$ there exists $h \in \mathcal{H}_n(D)$ such that $h(x_j) \in \Phi_j(x_j)$, $(1 \leqslant j \leqslant n+1)$.

Consider firstly $n+1$ distinct elements $x_1, x_2, \dots, x_{n+1} \in D$. By (ii) there exist $y_j \in \Phi_j(x_j)$ $(1 \leqslant j \leqslant n+1)$ such that

$$y_i = \sum_{\substack{j=1 \\ j \neq i}}^{n+1} c_j(x_i; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) y_j.$$

for some $i \in \{1, \dots, n+1\}$. Since $\mathcal{H}_n(D)$ is a vector space, the function

$$h(x) = \sum_{\substack{j=1 \\ j \neq i}}^{n+1} c_j(x; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) y_j$$

is an element of $\mathcal{H}_n(D)$ and satisfies

$$h(x_j) = y_j \in \Phi_j(x_j) \qquad (1 \leqslant j \leqslant n+1).$$

Suppose now that the points $x_1, x_2, \dots, x_{n+1} \in D$ are not distinct, for instance $x_1 = x_2 = \dots = x_{k_1} =: \bar{x}_1$, $x_{k_1+1} = x_{k_1+2} = \dots = x_{k_2} =: \bar{x}_2$, $\dots$, $x_{k_l+1} = x_{k_l+2} = \dots = x_{n+1} =: \bar{x}_{l+1}$. By (i) we can choose $y_1 \in \bigcap_{i=1}^{k_1} \Phi_i(\bar{x}_1)$, $y_2 \in \bigcap_{i=k_1+1}^{k_2} \Phi_i(\bar{x}_2)$, $\dots$, $y_{l+1} \in \bigcap_{i=k_l+1}^{n+1} \Phi_i(\bar{x}_{l+1})$. Clearly there exists at least one function $h \in \mathcal{H}_n(D)$ such that $h(\bar{x}_i) = y_i$ $(1 \leqslant i \leqslant l+1)$. Hence $h(x_i) \in \Phi_i(x_i)$ $(1 \leqslant i \leqslant n+1)$ and the proof is complete. □

4. Applications. The following result extends under many aspects Theorem 1 of [7].

Corollary 1. *Let $\mathcal{H}_n(D)$ be an n–dimensional Haar space and $F : D \longrightarrow \mathrm{cc}(\mathbb{R})$. The following statements are equivalent:*

(i) there exists an $h \in \mathcal{H}_n(D)$ such that $h(x) \in F(x)$ for all $x \in D$;

(ii) for any $n+1$ distinct elements $x_1, x_2, \dots, x_{n+1} \in D$ there exists an index $i \in \{1, 2, \dots, n+1\}$ such that

$$F(x_i) \cap \sum_{\substack{j=1 \\ j \neq i}}^{n+1} c_j(x_i; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) F(x_j) \neq \emptyset.$$

Proof. The implication $(i) \Rightarrow (ii)$ follows immediately from Lemma. To prove the converse we take $\Phi_i = F$, $i = 1, 2, \dots, n+1$. Then the condition of Theorem 1 are automatically fulfilled, hence the result follows from Theorem 1. □

Next we present the following generalization of Theorem 2 of [7].

Corollary 2. *Let $I \subset \mathbb{R}$ be an interval and $\mathcal{H}_n(I)$ an n–dimensional Haar space. Let $f_1, \dots, f_{n+1}, g_1, \dots, g_{n+1}$ be real functions defined on I with the following properties:*

(i) $\max\{f_1(x), \dots, f_{n+1}(x)\} \leqslant \min\{g_1(x), \dots, g_{n+1}(x)\}$ *for each $x \in I$;*

(ii) for any $x_1 < \dots < x_{n+1}$ belonging to I there exists an index $i \in \{1, \dots, n+1\}$ such that

$$\big[f_i(x_i), g_i(x_i)\big] \cap \sum_{\substack{j=1 \\ j \neq i}}^{n+1} c_j(x_i; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1})\big[f_j(x_j), g_j(x_j)\big] \neq \emptyset.$$

Then there exist $i_0 \in \{1, \dots, n+1\}$ and $h \in \mathcal{H}_n(I)$ such that $f_{i_0}(x) \leqslant h(x) \leqslant g_{i_0}(x)$ for all $x \in I$.

Proof. By (i) we get $f_i \leqslant g_i$ for all $i \in \{1, \dots, n+1\}$. Let $\Phi_i(x) = \big[f_i(x), g_i(x)\big]$, $x \in I$, $i = 1, \dots, n+1$. Applying (i) once again we obtain $\bigcap_{i=1}^{n+1} \Phi_i(x) \neq \emptyset$ for all $x \in I$. Then the multifunctions $\Phi_1, \dots, \Phi_{n+1} : I \longrightarrow \mathrm{cc}(\mathbb{R})$ fulfil all the assumptions of Theorem 1. Thus Theorem 1 can be applied to obtain the desired conclusion. □

5. Separation by polynomials. Let $I \subset \mathbb{R}$ be an interval and $\mathcal{P}_{n-1}(I)$ be the set of all polynomials $p : I \longrightarrow \mathbb{R}$ of degree at most $n-1$. Clearly, $\mathcal{P}_{n-1}(I)$ is an n–dimensional Haar space on I. According to Lagrange interpolation theorem, in this particular case $(\mathcal{H}_n(I) = \mathcal{P}_{n-1}(I))$ for any distinct $x_1, x_2, \dots, x_n \in I$ we have

$$c_j(x; x_1, x_2, \dots, x_n) = \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{x - x_k}{x_j - x_k}$$

With this significance for $c_j(x; x_1, x_2, \dots, x_n)$ we have another generalization of Theorem 2 of [7].

Corollary 3. *Let $f_1, \dots, f_{n+1}, g_1, \dots, g_{n+1}$ be real functions defined on I with the following properties:*

(i) $\max\{f_1(x), \dots, f_{n+1}(x)\} \leqslant \min\{g_1(x), \dots, g_{n+1}(x)\}$ *for each $x \in I$;*

(ii) for any $x_1 < \dots < x_{n+1}$ belonging to I there exists an index $i \in \{1, \dots, n+1\}$ such that

$$(1) \quad f_i(x_i) \leqslant \sum_{j \in S_1} c_j(x_i; x_1, \dots, x_{i-1}, x_{i+1} \dots, x_{n+1}) g_j(x_j) + \\ + \sum_{j \in S_2} c_j(x_i; x_1, \dots x_{i-1}, x_{i+1}, \dots, x_{n+1}) f_j(x_j)$$

and

$$(2)\quad g_i(x_i) \geqslant \sum_{j\in S_1} c_j(x_i; x_1, \ldots, x_{i-1}, x_{i+1} \ldots, x_{n+1}) f_j(x_j) +$$
$$+ \sum_{j\in S_2} c_j(x_i; x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_{n+1}) g_j(x_j)$$

where $S_1 = \{j : 1 \leqslant j \leqslant n+1,\ j-i \text{ is odd}\}$, $S_2 = \{j : 1 \leqslant j \leqslant n+1,\ j \neq i,\ j-i \text{ is even}\}$.

Then there exist $i_0 \in \{1, \ldots, n+1\}$ *and* $h \in \mathcal{P}_{n-1}(I)$ *such that* $f_{i_0}(x) \leqslant h(x) \leqslant g_{i_0}(x)$ *for all* $x \in I$.

Proof. If $x_1 < \cdots < x_{n+1}$ ($x_j \in I$) it is not hard to see that $c_j(x_i; x_1, \ldots, x_{i-1}, x_{i+1} \ldots, x_{n+1})$ is positive if $j \in S_1$ and negative if $j \in S_2$.

Let u and v be the right hand sides of the inequalities (1) and (2), respectively. Then $v \leqslant u$ and $[f_i(x_i), g_i(x_i)] \cap [v, u] \neq \emptyset$ (otherwise $g_i(x_i) < v$ contradicting (2), or $u < f_i(x_i)$ contradicting (1)).

It is easy to verify that

$$[v, u] = \sum_{\substack{j=1\\ j\neq i}}^{n+1} c_j(x_i; x_1, \ldots, x_{i-1}, x_{i+1} \ldots, x_{n+1}) \big[f_j(x_j), g_j(x_j)\big].$$

Hence the condition (ii) of Corollary 2 is satisfied. Thus the result follows from Corollary 2. □

Theorems on selections of multifunctions as well as theorems on separation by functions belonging to some classes are often useful for investigating the problems of the stability of Hyers-Ulam type (see [1], [5], [7]). Our last result is in this spirit.

Corollary 4. *Let* $\varphi \in \mathcal{P}_{n-1}(I)$ *and* $f_1, f_2, \ldots, f_{n+1} : I \longrightarrow \mathbb{R}$ *satisfy the following conditions:*

(i) $\big|f_i(x) - f_j(x)\big| \leqslant \varphi(x)$, *for each* $i, j \in \{1, 2, \ldots, n+1\}$ *and all* $x \in I$;

(ii) for any $x_1 < \cdots < x_{n+1}$ *belonging to* I *there exists an index* $i \in \{1, \ldots, n+1\}$ *such that*

$$(3)\qquad \Big|f_i(x_i) - \sum_{\substack{j=1\\ j\neq i}}^{n+1} c_j(x_i; x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1}) f_j(x_j)\Big| \leqslant \varphi(x_i).$$

Then there exists an $h \in \mathcal{P}_{n-1}$ *such that*

$$\big|f_i(x) - h(x)\big| \leqslant \frac{3}{2}\varphi(x)$$

for all $x \in I$, $i \in \{1, 2, \ldots, n+1\}$.

Proof. Let $g_i = f_i + \varphi$ $(1 \leqslant i \leqslant n+1)$. Then, by (i) we immediately get

$$\max\{f_1(x), \dots, f_{n+1}(x)\} \leqslant \min\{g_1(x), \dots, g_{n+1}(x)\}, \text{ for each } x \in I.$$

Fix $x_1, \dots, x_n \in I$ such that $x_1 < \dots < x_{n+1}$. Since $\varphi \in \mathcal{P}_{n-1}(I)$, by Lemma 1 it admits the representation

$$\varphi(x) = \sum_{\substack{j=1\\ j\neq i}}^{n+1} c_j(x; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1})\varphi(x_j).$$

Keeping in mind the fact that $c_j(x_i; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1})$ is positive if $j \in S_1$ and negative if $j \in S_2$, from (3) we succesively infer:

$$\begin{aligned} f_i(x_i) &\leqslant \sum_{\substack{j=1\\ j\neq i}}^{n+1} c_j(x_i; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) g_j(x_j) \leqslant \\ &\leqslant \sum_{j\in S_1} c_j(x_i; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) g_j(x_j) + \\ &+ \sum_{j\in S_2} c_j(x_i; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) f_j(x_j) \end{aligned}$$

and

$$\begin{aligned} g_i(x_i) &\geqslant \sum_{\substack{j=1\\ j\neq i}}^{n+1} c_j(x_i; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) f_j(x_j) \geqslant \\ &\geqslant \sum_{j\in S_1} c_j(x_i; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) f_j(x_j) + \\ &+ \sum_{j\in S_2} c_j(x_i; x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}) g_j(x_j). \end{aligned}$$

Therefore the functions f_i, g_i $(1 \leqslant i \leqslant n+1)$ satisfy the conditions of Corollary 3. Thus, for some $i_0 \in \{1, 2, \dots, n+1\}$ there exist $h' \in \mathcal{P}_{n-1}(I)$ such that $f_{i_0}(x) \leqslant h'(x) \leqslant g_{i_0}(x) = f_{i_0}(x) + \varphi(x)$. Let $h(x) = h'(x) - \frac{1}{2}\varphi(x)$. Hence $h \in \mathcal{P}_{n-1}(I)$ and $f_{i_0}(x) - \frac{1}{2}\varphi(x) \leqslant h(x) \leqslant f_{i_0}(x) + \frac{1}{2}\varphi(x)$, from which we obtain $|f_{i_0}(x) - h(x)| \leqslant \frac{1}{2}\varphi(x)$, $x \in I$. For $i \in \{1, 2, \dots, n+1\}$ and $x \in I$ by (i) we get

$$|f_i(x) - h(x)| \leqslant |f_i(x) - f_{i_0}(x)| + |f_{i_0}(x) - h(x)| \leqslant \varphi(x) + \frac{1}{2}\varphi(x) = \frac{3}{2}\varphi(x),$$

which finishes the proof. $\square$

References

[1] BALAJ, M., NIKODEM, K.: Remarks on Bárány's theorem and affine selections, *Discrete Math.* **224** (2000), 259–263.

[2] BEHRENDS, E., NIKODEM, K.: A selection theorem of Helly type and its applications, *Studia Math.* **116** (1995), 43–48.

[3] BORWEIN, P. B.: On Sylvester problem and Haar spaces, *Pacific J. of Math.* **109** (1983), 275–278.

[4] HOLMES, R. B.: A Course on Optimization and Best Approximation. Lecture Notes in Mathematics 257, *Springer–Verlag*, Berlin–Heidelberg–New York 1972.

[5] NIKODEM, K., WĄSOWICZ, SZ.: A sandwich theorem and Hyers-Ulam stability of affine functions, *Aequationes Math.* **49** (1995), 160–164.

[6] SINGER, I.: Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces, *Springer-Verlag*, Berlin-Neidelberg, 1970.

[7] WĄSOWICZ, SZ.: Polynomial selections and separation by polynomials, *Studia Math.* **120** (1996), 75–82.

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