ON THE STABILITY OF THE EQUATION STEMMING FROM LAGRANGE MVT

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ABSTRACT. The stability behaviour of the functional equation $F(y) - F(x) = (y - x)f\big((x+y)/2\big)$ is studied. It is proved that this equation is superstable i.e. if f, F satisfy $\big|F(y) - F(x)(y-x)f\big((x+y)/2\big)\big| \le \varepsilon$ then f satisfies this equation (with some F). In order to obtain this result the equation $h\Delta_h^2 f(x) = 0$ is considered and it is proved that also this equation is superstable.

1. INTRODUCTION

The Lagrange mean value theorem states that if a function $F : [x, y] \to \mathbb{R}$ is differentiable in the interval (x, y) and continuous in $[x, y]$ and $f = F'$ then there exists a point $z \in (x, y)$ such that $F(y) - F(x) = (y - x)f(z)$. However, it is easy to observe that if we consider the function $F(t) = t^2$, then the point z occurring in the above equation is equal to $\frac{x+y}{2}$ and therefore functions $F(t) := t^2, f(t) := 2t$ satisfy the following functional equation

$$F(y) - F(x) = (y - x)f\Big(\frac{x+y}{2}\Big). \tag{1.1}$$

J. Aczél in [1] solved this equation proving that all solutions are of the form $F(t) = at^2 + bt + c$, $f(t) = 2at + b$. A lot of generalizations of this result may be found in [3].

In this paper we investigate the Hyers–Ulam stability of the equation (1.1). Thus we consider the inequality

$$\Big|F(y) - F(x) - (y - x)f\Big(\frac{x+y}{2}\Big)\Big| \le \varepsilon, \tag{1.2}$$

where ε is some fixed positive number. We show in Theorem 3.2 that if (1.2) holds then f is necessarily an affine function of the form $f(x) = cx + b$ for some constants b, c. Our stability result, stated in Theorem 3.3, shows the superstability behaviour from the point of view of the function f. Namely if (1.2) holds, then equation (1.1) must be fulfilled, with $F(x)$ replaced by $G(x) := xf(x/2)$. It is also worth mentioning that one should not expect the stability behaviour for the function F since if inequality (1.2) is satisfied by some pair (F, f) then it is always satisfied by every function of the form $F(x) + c$, $c \in \mathbb{R}$ (and the same f).

To obtain this result we will need to study the stability of equation $h\Delta_h^2 f(x) = 0$ which has the same set of solution as $\Delta_h^2 f(x) = 0$ but as it will be shown the stability behaviour of these two equations is different.

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2. Stability of the equation $h\Delta_h^2 f(x) = 0$

Let $f:\mathbb{R}\to\mathbb{R}$. Recall that the difference operator is defined as

$$\Delta_h f(x) := f(x+h)-f(x),\quad \Delta_h^{n+1} f(x) := \Delta_h\big(\Delta_h^n f(x)\big),\quad x,h\in\mathbb{R},\ n\in\mathbb{N}.$$

Thus obviously $\Delta_h^2 f(x) = f(x+2h)-2f(x+h)+f(x)$, $x,h\in\mathbb{R}$.

Lemma 2.1. *Let $f:\mathbb{R}\to\mathbb{R}$, $\varepsilon>0$ and $M>0$. Assume that the inequality*

$$\big|\Delta_h^2 f(x)\big| \le \varepsilon \tag{2.1}$$

holds for all $x\in\mathbb{R}$, $h>M$. Then $\big|\Delta_h^2 f(x)\big|\le 2\varepsilon$ for all $x,h\in\mathbb{R}$.

Proof. Fix $x,h\in\mathbb{R}$ and take $H>M+|h|$. Obviously $H>M$, $H-h>M$ and $H+h>M$. Applying (2.1) to $x+2h-H$ and $H-h$ we get

$$\big|f(x+H)-2f(x+h)+f(x+2h-H)\big|\le\varepsilon,$$

while using (2.1) for $x+2h-H$ and H we obtain

$$\big|f(x+2h+H)-2f(x+2h)+f(x+2h-H)\big|\le\varepsilon.$$

Hence, by the triangle inequality,

$$\big|2f(x+2h)-2f(x+h)-f(x+2h+H)+f(x+H)\big|\le 2\varepsilon. \tag{2.2}$$

Now we use (2.1) for $x-H$ and $H+h$, whence

$$\big|f(x+2h+H)-2f(x+h)+f(x-H)\big|\le\varepsilon.$$

Finally, we apply (2.1) to $x-H$ and H:

$$\big|f(x+H)-2f(x)+f(x-H)\big|\le\varepsilon.$$

The triangle inequality implies

$$\big|-2f(x+h)+2f(x)+f(x+2h+H)-f(x+H)\big|\le 2\varepsilon.$$

The above inequality together with (2.2) gives

$$\big|2f(x+2h)-4f(x+h)+2f(x)\big|\le 4\varepsilon.$$

The Lemma follows by dividing this last inequality by 2. □

As a direct consequence of this theorem we obtain superstability of the equation

$$h\Delta_h^2 f(x)=0.$$

Corollary 2.2. *If a function $f:\mathbb{R}\to\mathbb{R}$ satisfies the inequality $\big|h\Delta_h^2 f(x)\big|\le\varepsilon$ for all $x,h\in\mathbb{R}$ and some $\varepsilon>0$, then there exist an additive function $a:\mathbb{R}\to\mathbb{R}$ and a constant $b\in\mathbb{R}$ such that $f(x)=a(x)+b$, $x\in\mathbb{R}$.*

Proof. For an arbitrary $\delta>0$ choose $M>0$ large enough to $\frac{\varepsilon}{M}<\frac{\delta}{2}$. Then for all $h>M$ we have $\big|\Delta_h^2 f(x)\big|\le\frac{\varepsilon}{h}<\frac{\varepsilon}{M}<\frac{\delta}{2}$. By Lemma 2.1 we infer that $\big|\Delta_h^2 f(x)\big|\le\delta$ for all $x,h\in\mathbb{R}$. Tending with δ to 0 we arrive at $\Delta_h^2 f(x)=0$, $x,h\in\mathbb{R}$, which is equivalent to the Jensen functional equation. Then we obtain the desired form of f (cf. [2, Theorem XIII.2.1, p. 315]). □

By Lemma 2.1 we can easily obtain also the following result, which may be interesting in its own right.

Corollary 2.3. *If $f : \mathbb{R} \to \mathbb{R}$ satisfies the Jensen functional equation*

$$f\Big(\frac{x+y}{2}\Big) = \frac{f(x)+f(y)}{2} \tag{2.3}$$

for all $x, y \in \mathbb{R}$ such that $|x-y| > M$ (where M is some positive number), then f satisfies (2.3) *for all $x, y \in \mathbb{R}$.*

Proof. Fix $x \in \mathbb{R}$. Let $h > \frac{M}{2}$ and $y = x + 2h$. Then $|y - x| > M$ and by (2.3)

$$2f(x+h) = f(x) + f(x+2h),$$

i.e. $\Delta_h^2 f(x) = 0$ for all $x \in \mathbb{R}$, $h > \frac{M}{2}$. Using Lemma 2.1 for an arbitrary $\varepsilon > 0$ we obtain $\Delta_h^2 f(x) = 0$ for an $x, h \in \mathbb{R}$, which concludes the proof. □

3. Stability of the equation (1.1)

Lemma 3.1. *If the functions $f, F : \mathbb{R} \to \mathbb{R}$ satisfy the inequality* (1.2) *for all $x, y \in \mathbb{R}$, then f satisfies the inequality $\big|h\Delta_h^2 f(x)\big| \le \frac{3}{2}\varepsilon$ for all $x, h \in \mathbb{R}$.*

Proof. Fix $x, h \in \mathbb{R}$. Take in (1.2) $x + h$ in place of y. Then

$$\Big|F(x+h) - F(x) - hf\Big(x + \frac{h}{2}\Big)\Big| \le \varepsilon.$$

Putting here $x + h$ instead of x we obtain

$$\Big|F(x+2h) - F(x+h) - hf\Big(x + \frac{3h}{2}\Big)\Big| \le \varepsilon.$$

By the above two inequalities

$$\Big|F(x+2h) - F(x) - h\Big[f\Big(x + \frac{3h}{2}\Big) + f\Big(x + \frac{h}{2}\Big)\Big]\Big| \le 2\varepsilon. \tag{3.1}$$

On the other hand, taking in (1.2) $x + 2h$ in the place of y we arrive at

$$\big|F(x+2h) - F(x) - 2hf(x+h)\big| \le \varepsilon.$$

Jointly with (3.1) this means that

$$\Big|h\Big[2f(x+h) - f\Big(x + \frac{3h}{2}\Big) - f\Big(x + \frac{h}{2}\Big)\Big]\Big| \le 3\varepsilon.$$

Finally, substituting here $2h$ instead of h we obtain

$$\big|2h\big[2f(x+2h) - f(x+3h) - f(x+h)\big]\big| \le 3\varepsilon$$

Replacing here x by $x - h$ and dividing both sides by 2 we get the desired inequality. □

Theorem 3.2. *If the functions $f, F : \mathbb{R} \to \mathbb{R}$ satisfy the inequality* (1.2)*, then there exist the constants $c, b \in \mathbb{R}$ such that $f(x) = cx + b$, $x \in \mathbb{R}$.*

Proof. From Lemma 3.1 and Corollary 2.2 we obtain that

$$f(x) = a(x) + b \tag{3.2}$$

for some additive function a and some constant $b \in \mathbb{R}$. Without loss of generality assume $F(0) = 0$ (it is a trivial observation that f, F fulfil (1.2) if and only if $f, F + C$ fulfil (1.2) for any constant $C \in \mathbb{R}$). Hence, putting in (1.2) $x = 0$, we obtain

$$\Big|F(y) - yf\Big(\frac{y}{2}\Big)\Big| \le \varepsilon, \quad y \in \mathbb{R},$$

which used in (1.2) gives

$$\Big|yf\Big(\frac{y}{2}\Big)-xf\Big(\frac{x}{2}\Big)-(y-x)f\Big(\frac{x+y}{2}\Big)\Big|\le 3\varepsilon.$$

Using here the form (3.2) we may write

$$\Big|\frac{1}{2}ya(y)-\frac{1}{2}xa(x)-\frac{1}{2}(y-x)\big[a(x)+a(y)\big]\Big|\le 3\varepsilon,$$

whence

$$\big|xa(y)-ya(x)\big|\le 6\varepsilon.$$

Now let $n\in\mathbb{N}$. Substituting above nx, ny for x, y, respectively, we obtain by additivity

$$\big|xa(y)-ya(x)\big|\le\frac{6\varepsilon}{n^2},$$

whence tending with n to infinity we get $ya(x)=xa(y)$, $x,y\in\mathbb{R}$. Thus for $y=1$ and $c=a(1)$ we have $a(x)=cx$, $x\in\mathbb{R}$. This, together with (3.2), concludes the proof. $\square$

As an immediate consequence we get the result concerning stability of the equation (1.1).

Theorem 3.3. *If the functions $f, F:\mathbb{R}\to\mathbb{R}$ satisfy the inequality* (1.2) *then f satisfies the equation*

$$G(y)-G(x)=(y-x)f\Big(\frac{x+y}{2}\Big).$$

(with $G(x)=xf(x/2)$).

Remark 3.4. As we mentioned in the Introduction, it is not possible to prove that the function G occurring in Theorem 3.3 is close to F. Nevertheless, applying (1.2) for $x=0$ we get

$$\big|F(y)-F(0)-G(y)\big|\le\varepsilon,$$

which means that G is "close to F up to a constant".

References

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